Topology II, lecture 1 9. 4. 2018

Substitution by E. Doblo this week

Overview

- Poincaré duality: if M is a closed mf., $\dim M = d$, oriented

$$\Rightarrow \quad H_{d-k}(M;\mathbb{Z}) \xrightarrow{\sim} H^{d-k}(M;\mathbb{Z}) \qquad \forall k = 0,\dots,d$$

$$x \longmapsto x \cap [M]$$

where $\cap : H_d(X) \otimes H^k(X) \longrightarrow H_{d-k}(X)$, $[M] \in H_d(M;\mathbb{Z})$ "orientation class"

Example. $\mathbb{C}P^n$ has homology: $\Rightarrow$ cohomology:

$$\begin{tikzcd}[column sep=small] & & & 0 & 1 & 2 & & 2n-1 & 2n & & & & \\ 0 & \dots & 0 & \mathbb{Z} \arrow[drrrrrr, <->] & 0 & \mathbb{Z} \arrow[drr, <->] & \dots & \mathbb{Z} \arrow[dll, <->] & 0 & \mathbb{Z} \arrow[dllllll, <->] & 0 & \dots & 0 & \dots \\ 0 & \dots & 0 & \mathbb{Z} & 0 & \mathbb{Z} & \dots & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \dots & 0 & \dots \end{tikzcd}$$

$\mathbb{R}P^2$ has homology: $\mathbb{Z} \quad 0 \quad \mathbb{Z}/2$

cohomology: $\mathbb{Z} \quad \mathbb{Z}/2 \quad 0$

$\Big\} \Rightarrow \mathbb{R}P^2$ is non-orientable

- Hurewicz theorem: relation b/w homology and homotopy

$$\begin{tikzcd} \tilde{H}_*(X) \arrow[d, "p_*"', shift right] & \simeq & \underbrace{H_*(*)}_{\mathbb{Z}\text{ incl. }0} \oplus \underbrace{\tilde{H}_*(X)}_{\ker p_*} \\ H_*(*) \arrow[u, "i_*"', shift right] & & \end{tikzcd}$$

i_* induced by a base pt. $x_0 \in X$: $\quad * \xrightarrow{i} X$, $\quad * \longmapsto x_0$

X is c-connected if $\pi_k X = 0 \quad \forall k < c$

Thm. (Hurewicz) There is a nat. map $h_* : \pi_* X \longrightarrow \tilde{H}_* X$ for every pointed X s.t. h_k is an iso $\forall k < c$ and surjective for $k = c+1$.

Example.

k	0	1	$\dots$	$n-1$	n		
$\pi_k S^n$	0	0	$\dots$	0	$\mathbb{Z}$ $\downarrow\cong$	?	
$\tilde{H}_k S^n$	0	0	$\dots$	0	$\mathbb{Z}$	0	$\dots$

I.e. we obtain $\pi_n S^n = \mathbb{Z}$ from $\tilde{H}_n S^n = \mathbb{Z}$ and the Hurewicz thm.

- **Fibrations.** nice $f : X \to Y$ which induce LES in htp groups

$$\dots \to \pi_2 Y \to \pi_1 F \to \pi_1 X \to \pi_1 Y \to \dots \quad \text{where} \quad F = f^{-1}(*)$$

These give tools for calculating π_*.

- **Eilenberg – MacLane spaces:** A ab. gp. There is an iso

$$H^n(X;A) \simeq \underset{\substack{\uparrow \\ \text{htp. classes of maps}}}{[X,\ K(A,n)]}$$

where $K(A,n)$ is determined by $\pi_k K(A,n) = \begin{cases} 0 & k \neq n \\ A & k = n \end{cases}$

Ex. $K(\mathbb{Z},1) = S^1$

$K(\mathbb{Z},2) = \mathbb{C}P^\infty$

Review of homology

Def. A chain cx of of gps is a sequence of Ab gps.

$$\dots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \dots \quad \text{s.t. the differentials}$$

are gp morphisms and $d_{n-1} \circ d_n = 0$, i.e. $\operatorname{im} d_n \subseteq \ker d_{n-1}$.

$H_{n-1}(C_*, d) := \ker d_{n-1} \big/ \operatorname{im} d_n$, measures failure of exactness.

Def. A cochain complex is a sequence of Ab. gps

$$\dots \leftarrow C^n \xleftarrow{d_{n-1}} C^{n-1} \xleftarrow{d_{n-2}} C^{n-2} \leftarrow \dots \quad \text{s.t.} \quad d^2 = 0.$$

$H^{n-1}(C^*, d) := \ker d_{n-1} / \operatorname{im} d_{n-2}$.

Rmk. If (C_*, d) is a chain complex, one can form a cochain cx.

$$\begin{tikzcd} \operatorname{Hom}(C_*, \mathbb{Z}): \dots & \operatorname{Hom}(C_n; \mathbb{Z}) \arrow[l] & \operatorname{Hom}(C_{n-1}, \mathbb{Z}) \arrow[l, "d^*"'] & \dots \arrow[l] \\ & f \circ d & f \arrow[l, mapsto] & \end{tikzcd}$$

This is indeed a cochain complex since $\forall f : f \circ d \circ d = f \circ 0 = 0$.

Topology II, lecture 1

$$\begin{tikzcd}
\text{Spaces} \arrow[r] \arrow[d] & \text{Chain cxs.} \arrow[r, "H_*"] \arrow[dd] \arrow[dr, phantom, "\not\circlearrowleft"] & \text{Ab} \\
\text{sSets} \arrow[ur] & & \\
& \text{Cochain cxs} \arrow[uur, "H^*"'] &
\end{tikzcd}$$

Def. Singular complex: for a space X, $C_*(X;\mathbb{Z}) := \overset{\substack{\text{free ab. gp.}\\ \swarrow}}{\mathbb{Z}}\left[\begin{matrix}\text{continuous maps}\\ \Delta^n \longrightarrow X\end{matrix}\right]$

$$= \bigoplus_{\sigma:\Delta^n\to X} \mathbb{Z}$$

with differential $d: C_n(X) \longrightarrow C_{n-1}(X)$

$$(\sigma:\Delta^n\to X) \longmapsto \sum_{i=0}^{n} (-1)^i \cdot \left(\Delta^{n-1} \xrightarrow{\delta^i} \Delta^n \xrightarrow{\sigma} X\right)$$

where $\delta^i : \Delta^{n-1} \hookrightarrow \Delta^n$

$$(0\le t_0,\dots,t_n\le 1) \mapsto (0\le t_0,\dots,t_{i-1},\overset{i}{0},t_i,\dots,t_n\le 1).$$

with $\sum t_i = 1$

$$H_*^{sing}(X;\mathbb{Z}) := H_*\left(C_*^{sing}(X;\mathbb{Z})\right)$$

$$H^*_{sing}(X;\mathbb{Z}) := H^*\left(\mathrm{Hom}\left(C_*^{sing}(X;\mathbb{Z}),\mathbb{Z}\right)\right)$$

Prop. $f: X \to Y$ htp equivalence $\Rightarrow$ $H_*(X;\mathbb{Z}) \xrightarrow[f_*]{\sim} H_*(Y;\mathbb{Z})$

and $H^*(X;\mathbb{Z}) \xleftarrow[f^*]{\sim} H^*(Y;\mathbb{Z})$.

Cellular homology

Def. A CW-complex is a space with a filtration of closed subspaces:

$$\emptyset \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \dots \subseteq X$$

s.t. $\bigcup_{i\geq 0} X^{(i)} = X$ and

$$\begin{tikzcd}
\coprod_{I_n} S^n \arrow[r] \arrow[d, "\coprod(\partial\,\mathrm{incl})"] & X^{(n)} \arrow[d, "\subseteq"] \\
\coprod_{I_{n+1}} D^{n-1} \arrow[r] & X^{(n+1)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

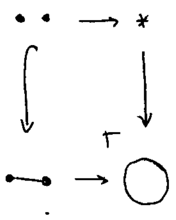
pushouts.

$X^{(n)}$: the n-skeleton

I_n : the set of n-cells

$\varphi_{e^{n+1}}: S^n \longrightarrow X^{(n)}$ attaching map of the $(n+1)$-cell e^{n+1}

Ex. S^1: $(S^1)^{(0)} = *$, $(S^1)^{(1)} = S^1$



S^n: $(S^n)^{(0)} = *$, $(S^n)^{(1)} = *, \dots, (S^n)^{(n-1)} = *$, $(S^n)^{(n)} = S^n$

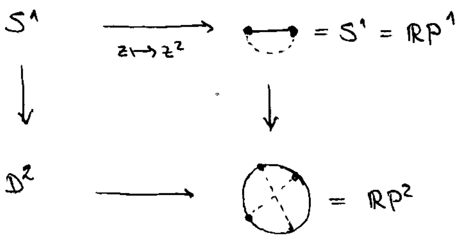
$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & * \arrow[d] \\ D^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^n \end{tikzcd}$$

$\mathbb{RP}^n$: one cell in each dimension, $(\mathbb{RP}^n)^{(k)} = \begin{cases} \mathbb{RP}^k & k \leq n \\ \mathbb{RP}^n & k \geq n \end{cases}$

with attaching maps

$$\begin{tikzcd} S^k \arrow[r] \arrow[d, hook] & \mathbb{RP}^k \arrow[d] \\ D^{k+1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{RP}^{k+1} \end{tikzcd}$$

For $k=1$:



Def. The cellular complex of X is

$$d: C_n^{cell}(X) \longrightarrow C_{n-1}^{cell}(X)$$
$$e \longmapsto \sum_{e^{n-1} \in I_{n-1}} \partial(e^n, e^{n-1})\, e^{n-1}$$

where $\partial(e^n, e^{n-1})$ is the degree of

$$S^{n-1} \xrightarrow[\varphi_{e^n}]{} X^{(n-1)} \twoheadrightarrow X^{(n-1)}/X^{(n-2)} \cong \bigvee_{e^{n-1} \in I_{n-1}} \overbrace{D^{n-1}/\partial}^{S^{n-1}} \xrightarrow[\text{proj}]{} S^{n-1}$$

where the degree of a map $f: S^{n-1} \longrightarrow S^{n-1}$ is the number d for which

$$\begin{tikzcd} f_*: H_{n-1}^{sing}(S^{n-1}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_{n-1}^{sing}(S^{n-1}) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z} \arrow[r, "\cdot d"] & \mathbb{Z} \end{tikzcd}$$

Def. $H_*^{cell}(X;\mathbb{Z}) := H_*(C_*^{cell}(X;\mathbb{Z}))$

$H^*_{cell}(X;\mathbb{Z}) := H^*(\mathrm{Hom}(C_*^{cell}(X;\mathbb{Z});\mathbb{Z}))$

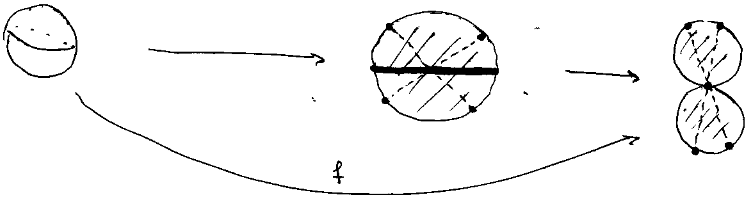
Ex. Calculating $H_*(\mathbb{R}P^n)$.

$$C_*^{cell}(\mathbb{R}P^n) = (\to 0 \to \underset{n}{\mathbb{Z}} \to \underset{n-1}{\mathbb{Z}} \to \dots \to \underset{1}{\mathbb{Z}} \to \underset{0}{\mathbb{Z}} \to 0)$$

We need to determine the degree of the map

$$S^k \xrightarrow[\varphi]{} \mathbb{R}P^k \twoheadrightarrow \mathbb{R}P^k/\mathbb{R}P^{k-1} \cong S^k$$

For $n = k = 2$: $S^2 \longrightarrow S^2/_{x\sim -x} \cong D^2/_{\substack{x\sim -x\\ \forall x\in\partial D^2}} \twoheadrightarrow (D^2/\dots)/(D^1/\dots) \cong S^2$



Claim: f is an iso on the upper and the lower hemisphere, and $f|_{upper} = f|_{lower} \circ$ antipodal map of S^k

$$\Rightarrow \deg f = 1 + (-1)^k = \begin{cases} 0 & 2\nmid k \\ 2 & 2\mid k \end{cases}$$

$$\Rightarrow \quad 2\nmid n: \quad 0 \longrightarrow \overset{n}{\mathbb{Z}} \xrightarrow{0} \overset{n-1}{\mathbb{Z}} \xrightarrow{2} \mathbb{Z} \longrightarrow \dots \xrightarrow{2} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \longrightarrow 0$$

$$2\mid n: \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow \dots \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

homology

$$\Rightarrow \quad 2\nmid n: \quad \dots 0 \quad \overset{n}{\mathbb{Z}} \quad \overset{n-1}{0} \quad \dots \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z} = H_*(\mathbb{R}P^n)$$

$$2\mid n: \quad \dots 0 \quad 0 \quad \mathbb{Z}/2 \quad \dots \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z} = H_*(\mathbb{R}P^n)$$

Thm. $H_*^{cell}(X) \cong H_*^{sing}(X)$, $H^*_{cell}(X) \cong H^*_{sing}(X)$

$H^*(X;A) := H^*(\mathrm{Hom}_{\mathbb{Z}}(C_*(X,\mathbb{Z}),A))$

11.4.2018

Universal coefficient theorem & Ext

Thm. (UCT) X space, A ab. gp. $\Rightarrow$ $\forall n > 0$ $\exists$ s.e.s.

$$0 \longrightarrow \operatorname{Ext}\left(H_{n-1}(X,\mathbb{Z}),A\right) \hookrightarrow H^n(X;A) \twoheadrightarrow \operatorname{Hom}_{\mathbb{Z}}\left(H_n(X;\mathbb{Z}),A\right) \longrightarrow 0$$

$$[f: C_n \to A] \longmapsto \left([x \in C_n] \mapsto f(x)\right)$$

where $\operatorname{Ext} = \operatorname{Ext}^1_{\mathbb{Z}}$, is the first Ext group over $\mathbb{Z}$.

Def. For a ring R, M a right R-module: M is projective if $\operatorname{Hom}_R(M,-)$ is left-exact.

Def. A projective / free resolution $P_* \to M$ of M is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $\forall P_i$ is projective / free.

Def. $\operatorname{Ext}^*_R(M,A) = H^*\Big(\cdots \leftarrow \underbrace{\operatorname{Hom}_R(P_1,A)}_{\text{degree } 1} \leftarrow \underbrace{\operatorname{Hom}_R(P_0,A)}_{\text{degree } 0} \leftarrow \underbrace{0}_{}\Big)$ where $P_* \to M$ proj. res.

we truncate the chain complex; if we didn't, this would be $\operatorname{Hom}_R(M,A)$.

Rmk.
- $\operatorname{Ext}^*_R(M,A)$ does not depend on P_*
- A projective module always has proj/free resolutions.
- For projective M: $\operatorname{Ext}^*_R(M,A) = \begin{cases} \operatorname{Hom}_R(M,A) & *=0 \\ 0 & \text{else} \end{cases}$
- In general, $\operatorname{Ext}^0_R(M,A) \cong \operatorname{Hom}_R(M,A)$
- For $R = \mathbb{Z}$, the R-modules are ab. gps.,

  $\operatorname{Ext}^*_{\mathbb{Z}}(M,A) = 0 \quad \forall * \geq 2$

  because one can always take the resolution

  $$0 \to \underbrace{\ker ev}_{\substack{\text{ab. subgroup of} \\ \text{a free ab.gp.} \Rightarrow \text{free}}} \longrightarrow \mathbb{Z}[M] \twoheadrightarrow M \longrightarrow 0$$

Ex. $\operatorname{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/n, A)$: we take the resolution

$$0 \longrightarrow \mathbb{Z} \xhookrightarrow{n\cdot} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n \longrightarrow 0 \qquad \text{free}$$

$$\begin{tikzcd} H^*\big(0 & \operatorname{Hom}(\mathbb{Z},A) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & \operatorname{Hom}(\mathbb{Z},A)\big) \arrow[l, "n\cdot"'] \arrow[d, phantom, "\cong" rotate=90] \\ & A & A \arrow[l, "n\cdot"] \end{tikzcd}$$

$$\Rightarrow \operatorname{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/n, A) = \begin{cases} \overset{n\text{-torsion}}{\{a \in A \mid na = 0\}} & * = 0 \\ A/nA & * = 1 \\ 0 & \text{else} \end{cases}$$

Topology II, lecture 2

Ex. $H_*(\mathbb{RP}^2;A) = \begin{cases} 0 & \text{else} \\ \mathbb{Z}/2 & *=1 \\ \mathbb{Z} & *=0,2 \end{cases}$ $\overset{\text{UCT}}{\Rightarrow}$ $H^*(\mathbb{RP}^2;A) = \begin{cases} 0 & \text{else} \\ A/2A & *=1 \\ A & *=0,2 \end{cases}$

$*=0$: $$0 \longrightarrow \underbrace{\text{Ext}(\underbrace{H_{-1}}_{0}, A)}_{0} \longrightarrow H^0(\mathbb{RP}^2;A) \longrightarrow \underbrace{\text{Hom}_{\mathbb{Z}}(\underbrace{H_0(\mathbb{RP}^2;\mathbb{Z})}_{\mathbb{Z}}, A)}_{A} \longrightarrow 0$$

$$\Rightarrow H^0(\mathbb{RP}^2;A) = A$$

$*=1$. $$0 \to \underbrace{\text{Ext}(\underbrace{H_0(\mathbb{RP}^2;\mathbb{Z})}_{\mathbb{Z}}, A)}_{0} \to H^1(\mathbb{RP}^2;A) \xrightarrow{\cong} \underbrace{\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, A)}_{\{a\in A \mid 2a=0\}} \longrightarrow 0$$

$$\to H^1(\mathbb{RP}^2;A) = \{a\in A \mid 2a=0\}$$

$*=2$. $$0 \to \underbrace{\text{Ext}(\mathbb{Z}/2, A)}_{A/2A} \xrightarrow{\cong} H^2(\mathbb{RP}^2;A) \longrightarrow \underbrace{\text{Hom}(0,A)}_{0} \longrightarrow 0$$

$$\to H^2(\mathbb{RP}^2;A) = A/2A$$

$H^*(\mathbb{RP}^2;A) = 0$ $\forall * \geq 3$ because $\mathbb{RP}^2$ is a 2-dimensional mf.

Group cohomology

G group, R ring

Def. Group-ring $\underline{R[G]} = \bigoplus_G R = \{\sum r_g g \mid r_g \neq 0$ for only fin. many $g\in G$, $r_g \in R\}$

with a multiplication $\left(\sum_g r_g\cdot g\right)\left(\sum_h s_h h\right) = \sum_{g,h} \underbrace{(r_g \cdot s_h)}_{\in R} \underbrace{(g\cdot h)}_{\in G}$.

Rmk. An $R[G]$-module is an R-module with a G-action by R-module maps.

R is an $R[G]$-module: $r'\cdot\left(\sum_g r_g \cdot g\right) = \sum_g (r'\cdot r_g)$

Def. The group cohomology of G with coeffs in R is $H^*(G,R)$:

$\underline{H^*(G,R)} := \text{Ext}^*_{R[G]}(R,R)$.

Goal: $H^*(G,R) = H^*(\text{some space}, R)$

Let BG be the classifying space of G.

$$\pi_*(BG) = \begin{cases} 0 & * \neq 1 \\ G & * = 1 \end{cases}$$

Recall a construction of BG:

$$BG = \left| \text{s.set with } n\text{-simplices } G^{\times n} \right|$$

with face maps d_i $G^{\times n} \longrightarrow G^{\times(n-1)}$

$$(g_1,\dots,g_n) \longmapsto \begin{cases} (g_2,\dots,g_n) & i = 0 \\ (g_1,\dots,g_{n-1}) & i = n \\ (g_1,\dots,g_{i-1}, g_i \cdot g_{i+1}, g_{i+2},\dots,g_n) & i = 1,\dots,n-1 \end{cases}$$

and degeneracy maps s_j: $G^{\times n} \longrightarrow G^{\times(n+1)}$

$$(g_1,\dots,g_n) \longmapsto (g_1,\dots,\overset{i}{1},\dots,g_n)$$

Thm. $H^*(G,R) \cong H^*(BG,R)$.

Pf: The idea is to find a chain complex that computes both cohomologies.

Take the universal cover of BG, denoted $G \to EG \to BG$.

EG is contractible and it has a free G-action and the quotient is BG.

Recall that a free action if $g \cdot x = x \Rightarrow g = 1$.

Consider $C_*(EG,R) = R[\mathrm{Map}(\Delta^n, EG)]$ (free R-module).

This is an $R[G]$-module.

G acts on a basis element $f: \Delta^n \longrightarrow EG$ by $g \cdot f: \Delta^n \xrightarrow{f} EG \xrightarrow{g\cdot} EG$

Consider the cochain complex $\mathrm{Hom}_{R[G]}(C_*(EG,R), R) =: C^*$

These are not cochains on EG; that would be the case if we used Hom_R.

$$C^* = \mathrm{Hom}_R(C_*(EG,R), R)$$

$\mathrm{Hom}_{R[G]}(-,-) = \mathrm{Hom}_R(-,-)^G$

$$\cong \mathrm{Hom}_R\left(C_*(EG,R)/_G , R \right)$$

R has trivial G-action, univ. prop. of the quotient

$$\cong \mathrm{Hom}_R\Big(C_*\big(\underbrace{EG/_G}_{BG}, R\big), R \Big)$$

the free module functor commutes with quotients

$\Rightarrow H^*(C^*;R) = H^*(BG,R)$, as desired.

(First actual proof of the semester, yay!)

Now show that $H^*(C^*) \cong H^*(G; R) = \mathrm{Ext}^*_{R[G]}(R, R)$. Then the theorem follows.

<u>Claim.</u> $C_*(EG, R)$ is a free resolution of R as an $R[G]$-module.

By def. of Ext, this clearly finishes the proof.

1) WTS $C_*(EG, R) = R[\mathrm{Map}(\Delta^n, EG)] = \bigoplus_{\mathrm{Map}(\Delta^n, EG)} R$ is a free $R[G]$-module

Let $E_0 \subseteq EG$ be a subset of orbit representatives of $G \curvearrowright EG$,

i.e. choose a section of $EG \longrightarrow EG/_G = BG$.

Let $M_0 \subseteq \mathrm{Map}(\Delta^n, EG)$, $M_0 := \{ f \mid f(0) \in E_0 \}$

By freeness $\forall f : \Delta^n \longrightarrow EG$ can be written as $f = g \cdot f_0$ for a unique $g \in G$:

$f(0) = g \cdot$ (rep. of the orbit of $f(0)$ in E_0),

uniqueness: $x = gy = g'y \implies g^{-1}g'y = y \implies g^{-1}g' = 1 \implies g = g'$.

$$\implies C_n(EG, R) = \bigoplus_{\mathrm{Map}(\Delta^n, EG)} R = \bigoplus_{f \in M_0} \underbrace{\bigoplus_{G} R}_{R[G]} = \bigoplus_{M_0} R[G]$$

2) WTS $C_*(EG, R)$ is a free resol., i.e.

$$\cdots \longrightarrow C_1(EG, R) \longrightarrow C_0(EG, R) \xrightarrow{EG \to *} R \longrightarrow 0 \quad \text{is exact.}$$

$$\sum_{\substack{\text{finite} \\ f : \Delta^n \to EG}} r_f \cdot f \longmapsto \sum_f r_f$$

Since $EG \simeq *$: $H_*(C_*(EG, R)) = \begin{cases} 0 & * \neq 0 \\ R & * = 0 \end{cases}$

$* \neq 0 \implies C_*$ is exact for $* > 1$

$* = 0$:

$$\begin{tikzcd}
C_1(EG, R) \arrow[r, "fd"'] & C_0(EG, R) \arrow[r, "\varepsilon"] \arrow[d, phantom, "\cong" rotate=90] & R \arrow[l, dashed, bend left, "?"', "r"] \\
& R \oplus \ker \varepsilon \arrow[ur, bend right, "proj"'] &
\end{tikzcd}$$

split exact

$$H_0 \cong R \implies \underbrace{C_*(EG, R) / \mathrm{im}\, d}_{\cong R \oplus \ker \varepsilon / \mathrm{im}\, d \;\cong\; R \oplus (\ker \varepsilon / \mathrm{im}\, d)} = R$$

$\Bigg\}$ $\ker \varepsilon / \mathrm{im}\, d = 0$

$\Downarrow$

$\mathrm{im}\, d = \ker \varepsilon$, exactness. ✓

Source of this proof: McLane: Homology, Ch. IV, Prop. 11.4 □

Cup product

X space, R ring $\longrightarrow$ $H^*(X, R)$ is a graded ring:

$$H^p(X;R) \otimes H^q(X;R) \xrightarrow{\cup} H^{p+q}(X;R)$$

Moreover, it is graded commutative, i.e. $x \cup y = (-1)^{|x|\cdot|y|}\, y \cup x$

Furthermore we have naturality, i.e. for $f: X \longrightarrow Y$ continuous,

f^* is a ring homomorphism.

The cup product on homology is def'd via a cup product on cochains $C^p(X;R) \otimes C^q(X;R) \xrightarrow{\cup} C^{p+q}(X;R)$ which satisfies the Leibniz rule: $d(x \cup y) = d(x) \cup y + (-1)^{|x|}\, x \cup d(y)$.

$$\big(f : \mathrm{Map}(\Delta^p, R) \to R\big) \otimes \big(g : \mathrm{Map}(\Delta^q, R) \longrightarrow R\big) \mapsto \big(\mathrm{Map}(\Delta^{p+q}, R) \longrightarrow R\big)$$

$$\big(\sigma : \Delta^{p+q} \to R\big) \longmapsto f\big(d^*_{front}\sigma\big) \cdot g\big(d^*_{back}\sigma\big)$$

where $d_{front} : \Delta^p \longrightarrow \Delta^{p+q} \xrightarrow{\sigma} X$

$$(i_0, \dots, i_p) \longmapsto (i_0, \dots, i_p, \underbrace{0, \dots, 0}_{q})$$

$$d_{back} : (i_0, \dots, i_p) \mapsto (0, \dots, 0, i_0, \dots)$$

16.4.2018

External products & the Künneth Theorem

I. Graded rings & exterior algebras

$\cup : H^n(Z, R) \underset{\mathbb{Z}}{\otimes} H^m(Z, R) \longrightarrow H^{n+m}(Z, R)$

Z: simplicial set

R: commutative ring

$\cup$ is associative (Top I)

$1 \in H^0(Z, R)$ is represented by $\mathrm{const}_1 : Z_0 \longrightarrow R$, $x \cup 1 = 1 \cup x = x$

$\cup$ is R-linear (later)

Def. Graded ring: a collection $\{A^n\}_{n\in\mathbb{Z}}$, the A^n are ab. gps. together with

- $\forall n, m \in \mathbb{Z}$: $A^n \underset{\mathbb{Z}}{\otimes} A^m \xrightarrow{\mu_{n,m}} A^{n+m}$ (to be denoted by $\cdot$ later) group homomorphism
- $\exists\ 1 \in A^0$

such that the following hold

Topology II, lecture 3

- **Associativity**:

$$\begin{tikzcd}
A^n \otimes A^m \otimes A^l \arrow[r, "\mu_{n,m} \otimes id"] \arrow[d, "id \otimes \mu_{m,l}"'] \arrow[dr, phantom, "\circlearrowright"] & A^{n+m} \otimes A^l \arrow[d, "\mu_{n+m,l}"] \\
A^n \otimes A^{m+l} \arrow[r, "\mu_{n,m+l}"] & A^{n+m+l}
\end{tikzcd}$$

- **Unitality**: $1 \in A^0 \Leftrightarrow \exists! \, \iota : \mathbb{Z} \longrightarrow A^0, \quad \iota(1) = 1$ (this is obvious)

$$\begin{tikzcd}
A^n \cong A^n \otimes \mathbb{Z} \arrow[r, "id \otimes \iota"] \arrow[dr, equal] & A^n \otimes A^0 \arrow[d, "\mu_{n,0}"] \\
& A^n
\end{tikzcd}$$

right unitality

$$\begin{tikzcd}
A^n \cong \mathbb{Z} \otimes A^n \arrow[r, "\iota \otimes id"] \arrow[dr, equal] & A^0 \otimes A^n \arrow[d, "\mu_{0,n}"] \\
& A^n
\end{tikzcd}$$

left unitality

Using the abuse-of-notation-like $x \cdot y := \mu_{n,m}(x \otimes y)$, these properties are just

- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
- $a \cdot 1 = a$ and $1 \cdot a = a$. (Note that a priori a graded ring might not be a ring.)

Ex. $\{H^n(Z,R)\}_{n \in \mathbb{Z}}$ is a graded ring with $\cup : H^n(Z,R) \otimes H^m(Z,R) \longrightarrow H^{n+m}(Z,R)$,
$1 \in H^0(Z,R)$.

$$H^*(Z,R) := \bigoplus_{n \in \mathbb{Z}} H^n(Z,R) \quad \text{with} \quad H^n(Z,R) = 0 \text{ for } n < 0$$

This $\oplus$ will inherit a ring structure, making it possible to talk about "the cohomology ring".

Given a gr. rg $\{A^n\}_{n \in \mathbb{Z}}$:

- $a \in A^n$ is said to have degree n
- A^n is called the n^{th} graded piece or n^{th} group of the graded rg.

Def. A homomorphism $f : \{A^n\}_{n \in \mathbb{Z}} \longrightarrow \{B^n\}_{n \in \mathbb{Z}}$ is a collection of ab.group homomorphisms $\{f^n : A^n \longrightarrow B^n\}_{n \in \mathbb{Z}}$ with the multiplicative property:

$\forall x \in A^n \ \forall y \in A^m \ \forall n,m \in \mathbb{Z}: \quad f^{n+m}(x \cdot y) = f^n(x) \cdot f^m(y)$. and $f^0(1) = 1$.

GRing is the in this way obtained category of graded rings.

Def. An internally graded ring is a ring $\left(S^*, \{S^n\}_{n\in\mathbb{Z}}\right)$

- $S^n \subseteq S^*$ additive subgroups $\forall n\in\mathbb{Z}$ s.t.

1) $S^n \cdot S^m = \{x\cdot y \mid x\in S^n,\ y\in S^m\} \subseteq S^{n+m}$

2) $1 \in S^0$

3) $\forall x\in S^*$ there is a unique decomposition $x = \sum_{n\in\mathbb{Z}} x_n$

with $x_n \in S^n$ $\forall n\in\mathbb{Z}$, and almost all x_n are zero.

$S^n \xhookrightarrow{i_n} S^*$ induce by the univ. prop. of direct sums $\bigoplus_{n\in\mathbb{Z}} S^n \xrightarrow{I} S^*$.

Thus 3) is equivalent to I being an isomorphism.

Def. A morphism of internally graded rings is a ring homomorphism

$f: S^* \longrightarrow \bar{S}^*$ such that $f(S^n) \subseteq \bar{S}^n$ $\forall n\in\mathbb{Z}$.

Again, the S^n are called n^{th} graded pieces or the subgroup of degree n elements.

IGRing := the category of internally graded rings.

rop. There is an equivalence of categories

$$\begin{tikzcd} \mathrm{GRing} \arrow[r, shift left, "t"] & \mathrm{IGRing} \arrow[l, shift left, "g"] \\ \{A^n\}_{n\in\mathbb{Z}} \arrow[r, mapsto] & \bigoplus_{n\in\mathbb{Z}} A_n = A^* \\ \{S^n\}_{n\in\mathbb{Z}} & S^* \arrow[l, mapsto] \end{tikzcd}$$

together with $S^n \otimes S^m \longrightarrow S^{n+m}$

$x\otimes y \longmapsto x\cdot y$ (in S)

$\forall x \in A^*$ can be written uniquely as $x = \sum_{\mathbb{Z}} x_n$, $x_n \in A^n$, almost all zero

For $x = \sum x_n$, $y = \sum y_n$ the product is

$$x\cdot y = \sum_{n,m\in\mathbb{Z}} x_n y_m = \sum_{k\in\mathbb{Z}} \sum_{n+m=k} x_n y_m .$$

Clearly we have $1 \in A^0 \subseteq A^*$ as a unital elt,

and we obtain an i. gr. rg. with A^n as the n^{th} graded piece.

Pf: $\{A^n\}_{n\in\mathbb{Z}} \longmapsto \left(A^* = \bigoplus_{\mathbb{Z}} A^n,\ \{A^n\}_{n\in\mathbb{Z}}\right) \longmapsto \{A^n\}_{n\in\mathbb{Z}}$

Topology II, lecture 3

$$(S^*, \{S^n\}_{n\in\mathbb{Z}}) \longmapsto \{S^n\}_{n\in\mathbb{Z}} \longmapsto \bigoplus_{\mathbb{Z}} S^n \xrightarrow[I]{\cong} S^*$$ $\square$

(This proof is not complete, one has to check multiplicativity too.) (exercise)

From now on, we will go back and forth between $A^* = \bigoplus_{n\in\mathbb{Z}} A^n$ and $A^* = \{A^n\}_{n\in\mathbb{Z}}$ without further notice.

Cor. $H^*(Z, R) = \bigoplus_{n\in\mathbb{Z}} H^n(Z, R)$ is a ring with $\cup$.

i.e. if $x_n \in H^n(Z, R)$, $y_n \in H^n(Z, R)$: $\left(\sum_n x_n\right) \cup \left(\sum_n y_n\right) = \sum_k \sum_{n+m=k} x_n \cup y_m$.
almost all zero $\square$

Elementary calculations

Want to compute $H^*(Z; R)$, mostly for $Z = \mathcal{S}(X)$ of a space X.

Shorthand: $H^*(X; R) = H^*(\mathcal{S}(X), R)$.

$H^*(pt; R)$:

$$H^n(pt, R) \cong \begin{cases} R & n=0 \\ 0 & \text{else} \end{cases}$$

$$R \xrightarrow{\cong} H^0(pt, R)$$

$$r \longmapsto (f_r : pt \longrightarrow R, \quad f_r(pt) = r.)$$

$[f_r] \cup [f_s] = [f_{rs}]$ This follows from the def. of $\cup$. (exercise)

$\Rightarrow$ $H^*(pt, R)$ as a graded ring $\cong R$ concentrated in degree 0:

$H^*(pt, R) = \bigoplus_{n\in\mathbb{Z}} H^n(pt, R)$, where all direct summands but $H^0(pt, R) \cong R$ are 0.

$f: X \longrightarrow Y$ cont map of spaces $\rightsquigarrow$ we get an induced map in cohom:

$H^n(Y; R) \xrightarrow{f^*} H^n(X; R)$ with $f^*(x \cup y) = f^*(x) \cup f^*(y)$

$\Rightarrow$ $H^*(Y; R) = \bigoplus_{\mathbb{Z}} H^n(Y; R) \xrightarrow{f^*} \bigoplus_{\mathbb{Z}} H^n(X; R) = H^*(X; R)$

is a ring homomorphism of integral graded rings.

Any top space X has a unique cont map $X \xrightarrow{\pi} pt$. This induces a ring homomorphism $H^*(pt; R) \xrightarrow{\pi^*} H^*(X; R)$.

$\forall n \geq 0$: $H^n(X,R)$ has a natural R-module structure:

$$\underbrace{H^0(pt, R)}_{\cong R} \longrightarrow \underbrace{H^0(X;R)}$$

this is a ring with the cup product $H^0 \otimes H^0 \xrightarrow{\cup} H^0$

$$\begin{tikzcd}
H^0(X,R) \otimes H^n(X,R) \arrow[r, "\cup"] & H^n(X,R) \\
R \otimes H^n(X,R) \arrow[u, "\pi^* \otimes id"] \arrow[ur] &
\end{tikzcd}$$

$r \in R$, $f \in \mathcal{S}_n(X) \longrightarrow R \Rightarrow (r \cdot f)(\sigma) = r \cdot (f(\sigma))$ (exercise)

<u>Prop.</u>

$$\begin{tikzcd}
H^n(X;R) \underset{\mathbb{Z}}{\otimes} H^m(X;R) \arrow[r, "\cup"] \arrow[d] & H^{n+m}(X,R) \\
H^n(X;R) \underset{R}{\otimes} H^m(X;R) \arrow[ur, dashed, "\cup"', "\exists!"] &
\end{tikzcd}$$

i.e. the cup product factors over R,
i.e. it is linear: $x_1 r \otimes x_2 = x_1 \otimes r x_2$

<u>Recall</u> the def of $\cup$: Z a simplicial set,

$[f] \in H^n(Z,R)$ is an R-module: $f: Z_n \longrightarrow R$ has

$r[f] = [rf]$, $(r \cdot f)(z) = r \cdot (f(z))$.

and for $[f] \in H^n(Z,R)$, $[g] \in H^m(Z,R)$:

$$f \cup g : Z_{n+m} \longrightarrow R$$
$$z \longmapsto f(d^*_{front}(z)) \cdot g(d^*_{back}(z))$$

where $[n] \xrightarrow{d_{front}} [n+m]$, $i \longmapsto i$

$[m] \xrightarrow{d_{back}} [n+m]$, $i \longmapsto n+i$ with $Z = \mathcal{S}(X)$.

$$\begin{tikzcd}
\sigma: \Delta^{n+m} \arrow[r] & X \\
\Delta^n \arrow[u, "d_{front}"] & \Delta^m \arrow[ul, "d_{back}"']
\end{tikzcd}$$

$d^*_{front}(\sigma) = \sigma \circ d_{front}$

$d^*_{back}(\sigma) = \sigma \circ d_{back}$

Pf of Prop: NTS $f \cup rg = fr \cup g$

$$(f \cup rg)(z) = f(d^*_{front}(z)) \cdot (rg)(d^*_{back}(z)) = f((d^*_{front}(z)) \cdot r) \cdot g(d^*_{back}(z)) \qquad \square$$

From now on, we will treat $H^n(Z,R)$ as an R-algebra instead of simply an R-module.

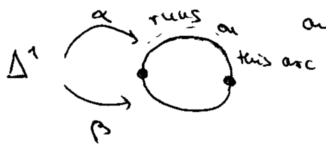
$H^*(S^1,R)$: $\quad H^n(S^1,R) \cong \begin{cases} R & n=0,1 \\ 0 & \text{else} \end{cases}$

$H^0(S^1,R) \cong R\cdot 1$ iso of R-modules

generators

$H^1(S^1,R) \cong R\cdot x$ iso of R-modules where $x \in H^1(S^1,R)$

Reminder: for $H_1(S^1,\mathbb{Z})$ we had



and $[\alpha-\beta]$ generated $H_1(S^1,\mathbb{Z})$

$$H_1(S^1,\mathbb{Z}) \longrightarrow H_1(S^1,R)$$

$\forall \alpha \in H^*(S^1,R) = \bigoplus_{\mathbb{Z}} H^n(S^1,R) \quad \exists!\, s,r \in R: \quad \alpha = r + sx$

Claim. $\alpha = r+sx$, $\beta = r'+s'x \Rightarrow \alpha\cdot\beta = rr' + (rs' + r's)x$, in particular, $x^2 = 0$.

$$x^2 = x \cup x \qquad H^1(S^1,R) \otimes H^1(S^1,R) \xrightarrow{\cup} \underbrace{H^2(S^1,R)}_{x^2 \in} = 0 \qquad \Rightarrow x^2 = 0$$

Cor. There is a ring iso:

$$H^*(S^1,R) \cong \underbrace{R[x]/(x^2)} \qquad |x| = 1 \text{ (degree 1)}$$

□

$$= \underbrace{R}_{\deg 0} \oplus \underbrace{R\cdot x}_{\deg 1}$$

The iso is not only of rings but of gr. rings as well.

$R[x]/(x^2)$ is the exterior algebra generated by x.

If $R = k$ field: $k[x]/(x^2) = \Lambda^*_k(V)$ where $\dim_k(V) = 1$.

Exterior algebras

18.4.2018

Def. $\Lambda_R[x_1,\dots,x_m]$ is an R-algebra with algebra generators $x_1,\dots,x_n$ subject to relations $x_i x_j = -x_j x_i \ \forall i \neq j$ and $x_i^2 = 0 \ \forall i$.

Prop. The additive basis for $\Lambda_R[x_1,\dots,x_m]$ is given by

$$\{x_{i_1} \cdots x_{i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq m\}$$

Pf: Use the relations to bring everything to a form like the sum of these (easy exercise).

□

$\Lambda_R[x_1,\dots,x_m]$ has a natural grading if one gives the x_i's appropriate degrees: $|x_i| = 2n_i - 1$ for some n_i

Thus the graded n^{th} piece of $\Lambda_R[x_1,\dots,x_m]$ will be given by the basis $\left\{x_{i_1}\cdots x_{i_k} \,\middle|\, 1\le i_1<\dots<i_k\le m,\ \sum_{\ell}\left(2n_{i_\ell}-1\right)=n\right\}$ and is denoted by $\Lambda^n_R[x_1,\dots,x_m]$. Note that it is finite dimensional.

$\Lambda_R = \bigoplus_n \Lambda^n_R$. Note that Λ_R is also finite dimensional since $\Lambda^n_R = 0$ for $n \gg 1$.

(The word 'natural' above just means 'automatic', it should not be understood as some category theoretic statement.)

Ex. $\Lambda_R[x,y] \ni a + bx + cy + dxy$

$|x|=1,\quad x^2=y^2=0,\quad |y|=3,\quad xy=-yx$

$\longrightarrow \Lambda_R[x,y] = \underset{\text{deg } 0}{R} \oplus \underset{\text{deg } 1}{R.x} \oplus \underset{\text{deg } 3}{R.y} \oplus \underset{\text{deg } 4}{R.x.y}$

Goal. $H^*\left(S^{2n_1-1}\times S^{2n_2-1}\times\dots\times S^{2n_k-1}, R\right) \cong \Lambda_R[x_1,\dots,x_k],\quad |x_i| = 2n_i-1$

In particular: $H^*(\underbrace{S^1\times\dots\times S^1}_{k}, R) \cong \Lambda_R[x_1,\dots,x_k]$ with $\forall |x_i|=1$.

Note: these do not hold for even dimensional spheres.

For this, we need the Künneth Theorem.

More elementary methods for computing $H^*(X,R)$

Let $\{Z_\alpha\}_{\alpha\in A}$ be a family of sets.

$$Z_\alpha \xhookrightarrow{i_\alpha} \coprod_{\alpha\in A} Z_\alpha \qquad \text{canonical inclusion}$$

$H^*\left(\coprod_\alpha Z_\alpha, R\right) \xrightarrow{i_\alpha^*} H^*(Z_\alpha, R)$ $\forall \alpha\in A$ is a ring homomorphism.

Univ. prop. of the product:

$$\begin{tikzcd}
H^*\left(\coprod Z_\alpha, R\right) \arrow[r, dashed, "\exists!\, c"] \arrow[dr, "i_\alpha^*"'] & \prod H^*(Z_\alpha, R) \arrow[d, "p_\alpha"] \arrow[dl, phantom, "\circlearrowright"] \\
& H^*(Z_\alpha, R)
\end{tikzcd} \qquad \forall \alpha\in A$$

Topology II, lecture 4

Prop. For a family of sets $\{Z_\alpha\}_{\alpha\in A}$ the canonical map

$$c:\ H^*(\coprod_\alpha Z_\alpha, R) \longrightarrow \prod_\alpha H^*(Z_\alpha, R)$$

is an isomorphism of rings.

Rmk. Note that $\bigoplus_n \prod_\alpha H^n(Z_\alpha, R) \xrightarrow{\not\cong} \prod_\alpha \bigoplus_n H^n(Z_\alpha, R)$ is <u>not</u> an isomorphism in general, just a canonical map.

In the category of graded rings, the product is the former, i.e. $\prod_\alpha H^*(Z_\alpha, R)$ is understood to be $\bigoplus_n \prod_\alpha H^n(Z_\alpha, R)$.

Notation. $\mathrm{Fun}(A,B)$ = functions from A to B (A, B sets)

PF OF PROP:

$$C^n(\coprod_\alpha Z_\alpha, R) = \mathrm{Fun}\left((\coprod_\alpha Z_\alpha)_n, R\right)$$

$$= \mathrm{Fun}\left(\coprod_\alpha Z_{\alpha,n}, R\right) \xrightarrow[\text{basic set theory}]{\cong} \prod_\alpha \mathrm{Fun}(Z_{\alpha,n}, R)$$

$$f \longmapsto (f|_{Z_{\alpha,n}})_{\alpha\in A}$$

$$\prod_\alpha \mathrm{Fun}(Z_{\alpha,n}, R) \cong \prod_\alpha C^n(Z_\alpha, R)$$

Thus we get an iso of chain cxs.:

$$C^*(\coprod_\alpha Z_\alpha, R) \xrightarrow{\cong} \prod_\alpha C^*(Z_\alpha, R)$$

After taking cohomology, we obtain

$$H^*(\coprod_\alpha Z_\alpha, R) \cong \prod_\alpha H^*(Z_\alpha, R). \qquad \square$$

Cor. $H^*(\underbrace{S^1 \amalg S^1 \amalg \ldots \amalg S^1}_{n}; R) \cong \Lambda_R[x_1] \times \ldots \times \Lambda_R[x_n]$ with $\forall |x_i| = 1$.
(Here $|(x_1, x_2, 0, \ldots, 0)| = 1$.)

Exercise. $S(\coprod_\alpha X_\alpha) \cong \coprod_\alpha S(X_\alpha)$.

Rmk. Note that $\Lambda_R[x_1] \times \ldots \times \Lambda_R[x_n]$ is a product of graded rings.
$R \times R$ is the deg 0, $R.x \times R.y$ is the deg 1 piece of $\Lambda_R[x] \times \Lambda_R[y]$, so $\Lambda_R[x] \times \Lambda_R[y] = (R\times R) \oplus (R.x \times R.y)$.

External product

$$H^n(X,R)\otimes_R H^m(Y,R) \xrightarrow{\times} H^{n+m}(X\times Y,R)$$

$$p_X : X\times Y \longrightarrow X, \qquad p_Y : X\times Y \longrightarrow Y$$

$$H^n(X,R) \xrightarrow{p_X^*} H^n(X\times Y,R), \qquad H^n(Y,R) \xrightarrow{p_Y^*} H^n(X\times Y,R)$$

$$\begin{tikzcd}
H^n(X,R)\otimes_R H^m(Y,R) \arrow[r, "p_X^*\otimes p_Y^*"] \arrow[dr, "\times", "\text{definition}"'] & H^n(X\times Y,R)\otimes_R H^m(X.Y,R) \arrow[d, "\cup"] \\
 & H^{n+m}(X\times Y,R)
\end{tikzcd}$$

The cross product $\times$ is defined by the above diagram. (external product)

Exercise. The following commutes:

$$\begin{tikzcd}
H^n(X,R)\otimes_R H^m(X,R) \arrow[r, "\times"] \arrow[dr, "\cup"'] & H^{n+m}(X\times X,R) \arrow[d, "\Delta^*"] \\
 & H^{n+m}(X,R)
\end{tikzcd}$$

where $\Delta : X \longrightarrow X\times X$ is the diagonal.

Recall $d_{\text{front}} : \Delta^n \longrightarrow \Delta^{n+m}$ $\quad (t_0,\dots,t_n)\longmapsto(t_0,\dots,t_n,0,\dots,0)$

$d_{\text{back}} : \Delta^m \longrightarrow \Delta^{n+m}$ $\quad (s_0,\dots,s_m)\longmapsto(0,\dots,0,s_1,\dots,s_m)$

$[f]\in H^n(X,R)$, $[g]\in H^m(Y,R)$ then their cross product $[f]\times[g]$ is repesented by

$f\times g\in C^{n+m}(X\times Y,R)$, $\forall\, \sigma=(\sigma_1,\sigma_2)\in \Delta^{n+m}\longrightarrow X\times Y$

$$(f\times g)(\sigma_1,\sigma_2) = f(\sigma_1\circ d_{\text{front}})\cdot g(\sigma_2\circ d_{\text{back}})$$

One still needs to check that this works.

One could also introduce the ext. product for ssets.

Topology II, lecture 4

Tensor product of graded rings

Let A^*, B^* be graded rings. (graded R-algebras).

$$(A^* \underset{R}{\otimes} B^*)^\ell = \bigoplus_{\substack{n,m \\ n+m=\ell}} A^n \underset{R}{\otimes} B^m \qquad \text{where} \quad A^* = \bigoplus_n A^n, \quad B^* = \bigoplus_m B^m$$

$$\underline{A^* \underset{R}{\otimes} B^*} \cong \left(\bigoplus_n A^n\right) \underset{R}{\otimes} \left(\bigoplus_m B^m\right) \cong \bigoplus_{n,m} \left(A^n \underset{R}{\otimes} B^m\right) = \bigoplus_n \bigoplus_m \left(A^n \underset{R}{\otimes} B^m\right)$$

$$\begin{array}{ccccc} (A^* \underset{R}{\otimes} B^*)^{\ell_1} & \otimes & (A^* \underset{R}{\otimes} B^*)^{\ell_2} & \longrightarrow & (A \underset{R}{\otimes} B)^{\ell_1+\ell_2} \\ (a \otimes b) & \otimes & (c \otimes d) & \longmapsto & (-1)^{|b| \cdot |c|} (ac \otimes bd) \\ |a|+|b|=\ell_1 & & |c|+|d|=\ell_2 & & \end{array}$$

Suppose A^*, B^*, C^* are gr rings. Then giving a graded ring homomorphism

$$A^* \underset{R}{\otimes} B^* \xrightarrow{f} C^*$$

is equivalent to giving abelian group homomorphisms

$$A^n \underset{R}{\otimes} B^m \xrightarrow{f_{n,m}} C^{n+m}$$

such that $(-1)^{|b||c|} f_{|a|+|c|,\, |b|+|d|} (ac \otimes bd) = f_{|a|,|b|}(a \otimes b) \cdot f_{|c|,|d|}(c \otimes d)$ $(\#)$ holds.

<u>Prop.</u> The exterior product

$$\times : \quad H^n(X,R) \underset{R}{\otimes} H^m(Y,R) \longrightarrow H^{n+m}(X \times Y, R) \qquad (n,m \geq 0)$$

defines a homomorphism of graded rings

$$\times : \quad H^*(X,R) \underset{R}{\otimes} H^*(Y,R) \longrightarrow H^*(X \times Y, R)$$

Pf: We need to check that $(\#)$ holds.

$$(a \cup c) \times (b \cup d) = p_X^*(a \cup c) \cup p_Y^*(b \cup d) =$$

$$= p_X^*(a) \cup p_X^*(c) \cup p_Y^*(b) \cup p_Y^*(d) =$$

$$= (-1)^{|b| \cdot |c|} p_X^*(a) \cup p_Y^*(b) \cup p_X^*(c) \cup p_Y^*(d) =$$

here the exponent should be $\underbrace{|p_X^*(c)|}_{=|c|} \cdot \underbrace{|p_Y^*(b)|}_{=|b|}$

$$= (-1)^{|b| \cdot |c|} (a \times b) \cup (c \times d).$$

□

<u>Thm.</u> (Künneth) X, Y CW-complexes such that $H^*(Y,R)$ is a flat R-module.

Then the external product

$$H^*(X,R) \underset{R}{\otimes} H^*(Y,R) \longrightarrow H^*(X \times Y, R)$$

is an isomorphism of graded rings.

This will take quite long to prove.

$$H^n(X\times Y, R) \cong \bigoplus_{\substack{k,l \\ k+l=n}} H^k(X,R) \underset{R}{\otimes} H^l(Y,R)$$ is a particular consequence of the above Thm.

There is a more general version for non-flat modules, but then the statement will involve Tor groups.

<u>Relative cup product</u>

X CW-complex, $A, B \subseteq X$ subcomplexes

Then there is a <u>relative cup product</u>

$$\underline{\cup}: \ H^n(X,A,R) \underset{R}{\otimes} H^m(X,B,R) \longrightarrow H^{n+m}(X, A\cup B, R).$$

$[f] \in H^n(X,A,R)$, $[g] \in H^m(X,B,R)$. Then we have representatives

$f: C_n(X,\mathbb{Z}) \longrightarrow R$, $\quad g: C_m(X,\mathbb{Z}) \longrightarrow R$ such that

$f|_{C_n(A,\mathbb{Z})} = 0$, $g\cancel{f}|_{C_m(B,\mathbb{Z})} = 0$.

Let $\sigma \in S_{n+m}(X)$.

$\underline{(f\cup g)(\sigma)} := f(d^*_{front}\,\sigma) \cdot g(d^*_{back}\,\sigma)$

If $\operatorname{im}\sigma \subseteq A$ or $\operatorname{im}\sigma \subseteq B$, then $(f\cup g)(\sigma) = 0$.

$\Rightarrow$ $f\cup g: C_{n+m}(X,\mathbb{Z}) \longrightarrow R$ vanishes on $C_{n+m}(A,\mathbb{Z}) + C_{n+m}(B,\mathbb{Z})$

$\Rightarrow$ $f\cup g: C_{n+m}(X,\mathbb{Z}) \big/ C_{n+m}(A,\mathbb{Z}) + C_{n+m}(B,\mathbb{Z}) \longrightarrow R$ (descends)

$C_*(A,\mathbb{Z}) + C_*(B,\mathbb{Z}) \subseteq C_*(X,\mathbb{Z})$ is a sub-chain complex

Thus one can define the following cochain complexes:

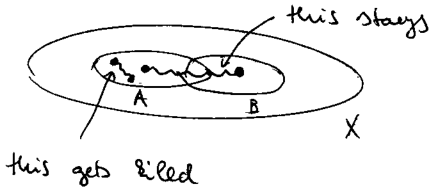
1) $C^*(A+B,R) := \mathrm{Hom}_{\mathbb{Z}}(C_*(A,\mathbb{Z}) + C_*(B,\mathbb{Z}), R)$

2) $C^*(X, A+B, R) := \mathrm{Hom}_{\mathbb{Z}}(C_*(X,\mathbb{Z}) / C_*(A,\mathbb{Z}) + C_*(B,\mathbb{Z}), R)$

$[f] \cup [g] \in H^{n+m}(X, A+B, R) = H^{n+m}(C^*(X, A+B, R))$

Note that in 2) we kill all chains that are in A, and those that are in B. But chains that are in A and B (and not in $A \cap B$) will stay.

23.04.2018



$C_*(A,\mathbb{Z}) + C_*(B,\mathbb{Z}) \subseteq C_*(A \cup B, \mathbb{Z})$ subchain complex

There is a natural map of chain complexes; the "restriction".

Apply $\mathrm{Hom}_{\mathbb{Z}}(-, R)$ to:

$$C^*(A \cup B, R) \xrightarrow[\text{chain map}]{} C^*(A+B, R)$$

$$C_*(X,\mathbb{Z}) / C_*(A,\mathbb{Z}) + C_*(B,\mathbb{Z}) \longrightarrow C_*(X,\mathbb{Z}) / C_*(A \cup B, \mathbb{Z})$$

Thus we obtain:

$$C^*(X, A \cup B, R) \longrightarrow C^*(X, A+B, R)$$

Moreover we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^*(X, A\cup B, R) \arrow[r] \arrow[d, "\simeq"] \arrow[rd, phantom, "\circlearrowright"] & C^*(X,R) \arrow[r] \arrow[d, equal] \arrow[rd, phantom, "\circlearrowright"] & C^*(A\cup B, R) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & C^*(X, A+B, R) \arrow[r] & C^*(X,R) \arrow[r] & C^*(A+B, R) \arrow[r] & 0
\end{tikzcd}$$

(The commutativity is basically tautological, f gets sent to f.)

<u>Goal:</u> the left and right vertical maps induce isos in H^*.

Case 1. X CW, A, B sub-CW

Case 2. X space, A, B open

(The statement is false for completely general X, A, B.)

$$C_*(A, \mathbb{Z}) + C_*(B, \mathbb{Z}) \subseteq C_*(A \cup B, \mathbb{Z})$$

By the lemma about small simplices, this inclusion induces iso on H_*.

UCT ⇒ $C^*(A \cup B, R) \longrightarrow C^*(A+B, R)$ gives an iso $H^*(A \cup B, R) \xrightarrow{\cong} H^*(A+B, R)$

By the naturality of the long exact cohomology sequence for cochain complexes we get: $C^*(X, A \cup B, R) \xrightarrow{\simeq} C^*(X, A+B, R)$ is a quasi-iso, i.e. $H^*(X, A \cup B, R) \xrightarrow{\cong} H^*(X, A+B, R)$ is an iso. □

$$\begin{tikzcd}
\text{Consequence: } \exists\ [f] \cup [g] \in & H^{n+m}(X, A+B, R) \\
\exists!\ [f] \cup [g] \in \arrow[u] & H^{n+m}(X, A \cup B, R) \arrow[u, "\cong"']
\end{tikzcd}$$

As a special case we get: $(B = \emptyset)$

$$H^n(X, A, R) \otimes H^m(X, R) \longrightarrow H^{n+m}(X, A, R).$$

This one can do without all the above stuff, in a direct way. (Easy exercise.)

We also have an external version of this product:

$$H^n(X, A, R) \otimes H^m(Y, B, R) \xrightarrow{\times} H^{n+m}(X \times Y, A \times Y \cup X \times B, R)$$

This generalises our previous $\times$.

$$\begin{tikzcd}
H^n(X, A, R) \otimes H^m(Y, B, R) \arrow[d, "p_X^* \otimes p_Y^*"] \arrow[dd, bend left=60, "\times"] \\
H^n(X \times Y, A \times Y, R) \otimes H^m(X \times Y, X \times B, R) \arrow[d, "\cup"] \\
H^{n+m}(X \times Y, A \times Y \cup X \times B, R)
\end{tikzcd}$$

Here we assume A and B to be open OR X CW, A, B sub-CW.

The product in the latter case is the product of CW complexes.

Special case: $B = \emptyset$

$$H^n(X,A,R) \otimes H^m(Y,R) \longrightarrow H^{n+m}(X\times Y,\ A\times Y,\ R)$$

Note that in the above discussion $\otimes$ always means $\otimes_R$.

We could also use $\otimes_{\mathbb{Z}}$, but that would factor over R, so that would be pointless.

$H^*(X,R) \otimes H^*(Y,R) \xrightarrow{\times} H^*(X\times Y, R)$ is a ring homomorphism

In fact, the above discussion gives an even more general ring hom:

$$H^*(X,A,R) \underset{R}{\otimes} H^*(Y,R) \xrightarrow{\times} H^*(X\times Y,\ A\times Y,\ R)$$

I.e. additively for a fixed $n \geq 0$ this map is given by

$$\underbrace{\bigoplus_{i\in\mathbb{Z}}}\ H^i(X,A,R) \underset{R}{\otimes} H^{n-i}(Y,R) \xrightarrow{\times} H^n(X\times Y,\ A\times Y,\ R)$$

we could write $i \geq 0$ instead as negative cohomologies are 0.

Thm. (Künneth) i) X, Y CW complexes, $A \subseteq X$ subcomplex and $\forall l$: $H^l(Y,R)$ is a finitely generated projective module. Then

$H^*(X,A,R) \underset{R}{\otimes} H^n(Y,R) \xrightarrow[\cong]{\times} H^*(X\times Y,\ A\times Y,\ R)$ is a ring iso.

ii) X finite CW, $H^l(Y,R)$ flat $\forall l$. Then

$$H^*(X,A,R) \underset{R}{\otimes} H^*(Y,R) \xrightarrow[\cong]{\times} H^*(X\times Y,\ A\times Y,\ R).$$

Note that the previous formulation of this statement (in lec. 4) was wrong.

Rmk. we only care about PIDs here actually. Before the proof we introduce some other stuff.

Notation. $$h^n(X,A) := \bigoplus_{i\in\mathbb{Z}} H^i(X,A,R) \underset{R}{\otimes} H^{n-i}(Y,R)$$

$$k^n(X,A) := H^n(X\times Y,\ A\times Y,\ R)$$

h^n, k^n are functors $\text{CW-pairs}^{op} \longrightarrow Ab$

Objects of CW-pairs: X CW, $A \subseteq X$ sub-CW.

Morphisms: $f : (X,A) \longrightarrow (Z,B)$ cont map with $f(A) \subseteq B$.

Homotopy in CW-pairs b/w f and g: $f \simeq g$

if $f, g: (X, A) \longrightarrow (Z, B)$ morphisms

then $H: X \times I \longrightarrow Z$ is a homotopy

if $H|_{X\times 0} = f$, $H|_{X\times 1} = g$ and $H(A \times I) \subseteq B$.

Def. A <u>(generalised) cohomology theory</u> is the following data:

- $\forall n \in \mathbb{Z}$ there is a contravar. functor E^n: CW-pairs$^{op} \longrightarrow Ab$, i.e. $E^n(X, A)$ for every $(X, A) \in$ Ob CW-pairs. $E^n(Z) := E^n(Z, \emptyset)$.
- $\forall n \in \mathbb{Z}$ there are connecting homomorphisms $\partial: E^n(A) \longrightarrow E^{n+1}(X, A)$ for every CW-pair (X, A), and these ∂ are natural, i.e. $\forall f: (X, A) \longrightarrow (Z, B)$ the diagram

$$\begin{tikzcd}
E^n(A) \arrow[r, "\partial"] & E^{n+1}(X, A) \\
E^n(B) \arrow[u, "f^*"] \arrow[r, "\partial"] \arrow[ur, phantom, "\circlearrowleft"] & E^{n+1}(Z, B) \arrow[u, "f^* = E^{n+1}(f)"']
\end{tikzcd}$$

commutes.

These data are subject to the following conditions:

1) Htp invariance: $f, g: (X, A) \longrightarrow (Z, B)$ and $f \simeq g \Rightarrow E^n(f) = E^n(g)$ $\forall n \in \mathbb{Z}$.

2) Long exact sequence: for any (X, A) the following is exact:

$$\ldots \longrightarrow E^n(X) \xrightarrow{i^*} E^n(A) \xrightarrow{\partial} E^{n+1}(X, A) \xrightarrow{j^*} E^{n+1}(X) \xrightarrow{i^*} E^{n+1}(A) \longrightarrow \ldots$$

where $i: A \hookrightarrow X$, $j: (X, \emptyset) \hookrightarrow (X, A)$ inclusions.

3) Excision: $\forall (X, A)$ the natural map

$$E^n(X/A, *) \longrightarrow E^n(X, A)$$

is an isomorphism. (Induced by the nat. projection $(X, A) \longrightarrow (X/A, *)$. For $A = \emptyset$, $X/A = X \amalg *$ by convention.)

4) Disjoint union axiom:

$$E^n\left(\coprod_\alpha X_\alpha\right) \xrightarrow{\cong} \prod_\alpha E^n(X_\alpha)$$

(The relative version will follow from the other axioms.)

This is the beginning of stable hom. theory.

<u>Notation.</u> All the above will be denoted by E^* for brevity.

Topology II, lecture 5-6

<u>Ex.</u> $\{h^n\}_{n\in\mathbb{Z}}$ and $\{k^n\}_{n\in\mathbb{Z}}$ form generalised homology theories. (see below)

<u>Rmk.</u> We do not ask for $E^n(pt) = 0$ unless $n = 0$.

<u>Ex.</u> $H^*(-, M)$ with M being abelian. This is ordinary homology because it satisfies the condition of the Rmk.

<u>Ex.</u> K-theory (topological): $K^*(-)$, $K^n(pt) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$
(complex)

<u>Ex.</u> $\Omega^*(-)$ cobordism

<u>Def.</u> A natural transformation b/w cohomology theories E^*, F^* is a set of nat. transformations $E^n \xrightarrow{\alpha^n} F^n$ which are compatible with the conn. morphisms:

$$\begin{tikzcd} E^n(A) \arrow[r, "\partial"] \arrow[d, "\alpha^n"'] \arrow[dr, phantom, "\circlearrowright"] & E^{n+1}(X,A) \arrow[d, "\alpha^{n+1}"] \\ F^n(A) \arrow[r, "\partial"] & F^{n+1}(X,A) \end{tikzcd}$$

$\forall (X,A)$.

This is denoted by $\alpha : E^* \longrightarrow F^*$.

<u>Prop.</u> $h^n(X,A) := \bigoplus_{i\in\mathbb{Z}} H^i(X,A,R) \otimes_R H^{n-i}(Y,R)$. Conn. homomorphisms come from the ones for sing. homology.

$k^n(X,A) := H^n(X\times Y, A\times Y, R)$ Suppose $\forall H^l(Y,R)$ are projective, f.gen. $\forall l \in \mathbb{Z}$.

The h^* and k^* are generalised cohomology theories and $\times : h^* \longrightarrow k^*$ is a nat. trf. of coh. theories.

Pf: First we deal with h^*.

- Homotopy invariance: h^n is a functor, singular homology is htp. invariant, $\oplus$ and $\otimes$ preserve equality. ✓
- LES: $\forall i \in \mathbb{Z}$:

$$\to H^{i-1}(A,R) \longrightarrow H^i(X,A,R) \longrightarrow H^i(X,R) \longrightarrow H^i(A,R) \longrightarrow H^{i+1}(X,A,R) \longrightarrow \dots$$

$H^{n-i}(Y,R)$ is f.gen. proj. R-module $\Rightarrow$ flatness $\Rightarrow$ preserves exactness with $\otimes$. So tensoring the above sequence yields an exact seq.

$$\to \bigoplus_i H^{i-1}(A,R) \otimes_R H^{n-i}(Y,R) \to \bigoplus_i H^i(X,A,R) \otimes H^{n-i}(Y,R) \to \bigoplus_i H^i(X,R) \otimes H^{n-i}(Y,R) \to$$

$$\to \bigoplus_i H^i(A,R) \otimes_R H^{n-i}(Y,R) \to \bigoplus_i H^{i+1}(X,A,R) \otimes H^{(n+1)-(i+1)}(Y,R) \to \dots$$

Take $\bigoplus_i$ $\longrightarrow$ LES for h^*. ✓
preserves exactness

- Excision: follows trivially from $H^k(X,A;R) \xleftarrow{\cong} H^k(X/A, *, R)$.
- Disj. union: now we will use f.g. projectiveness (so far we have only used flatness). NTS: $h^n\left(\coprod_\alpha X_\alpha\right) \xrightarrow{\cong} \coprod_\alpha h^n(X_\alpha)$

We know that $H^n\left(\coprod_\alpha X_\alpha, R\right) \xrightarrow{\cong} \prod_\alpha H^n(X_\alpha, R)$.

Infinite products and tensoring do not commute in general, not even for flat modules.

<u>Lemma</u> (algebra). N fin. generated projective R-module, then $\forall \{M_\alpha\}_\alpha$:

$$\left(\prod_\alpha M_\alpha\right) \otimes_R N \xrightarrow{\cong} \prod_\alpha \left(M_\alpha \otimes_R N\right).$$ □

$$\Rightarrow \begin{tikzcd} H^i\left(\coprod_\alpha X_\alpha, R\right) \otimes H^{n-i}(Y,R) \arrow[r, "\cong"] \arrow[dr, bend right, "\cong"'] & \left(\prod_\alpha H^i(X_\alpha,R)\right) \otimes H^{n-i}(Y,R) \arrow[d, "\cong"] \\ & \prod_\alpha \left(H^i(X_\alpha,R) \otimes H^{n-i}(Y,R)\right) \end{tikzcd}$$

Take $\oplus$ ⟶ ✓

k^* is also clearly a coh. theory: if (X,A) is a CW-pair then so is $(X\times Y, A\times Y)$, and use sing. homology. (So k^* does not even need any extra conditions like projectiveness or flatness.) ✓

25.04.2018

Recall that $\times$ is given by $H^i(X,A;R) \otimes H^{n-i}(Y,R) \longrightarrow H^n(X\times Y, A\times Y, R)$ (#)

$[\varphi] \quad [\psi] \longmapsto p_X^*([\varphi]) \cup p_Y^*([\psi])$

Goal: $\times$ is a nat trf. of gen coh. ths.

Enough: check that (#) is natural and compat. with ∂

Notation: $f: X \longrightarrow Z$, $f^*: H^i(Z,R) \longrightarrow H^i(X,R)$, $f^*: C^i(Z,R) \longrightarrow C^i(X,R)$

$\cup: H^i \otimes H^j \longrightarrow H^{i+j}$, $\cup: C^i \otimes C^j \longrightarrow C^{i+j}$

Note the abuse of notation and keep in mind that one is induced by the other.

$C_i(X,Z) \xrightarrow{d} C_{i-1}(X,Z)$ d chain diff

$C^i(X,Z) \xrightarrow{d} C^{i+1}(X,Z)$ d cochain diff.

Naturality of $\times$: let $f: (X, A) \longrightarrow (Z, B)$.

NTS commutativity of

$$\begin{tikzcd}
H^i(Z, B, R) \underset{R}{\otimes} H^{n-i}(Y, R) \arrow[r, "\times"] \arrow[d, "f^* \otimes \mathrm{id}"] & H^n(Z \times Y, B \times Y, R) \arrow[d, "(f \times \mathrm{id})^*"] \\
H^i(X, A, R) \underset{R}{\otimes} H^{n-i}(Y, R) \arrow[r, "\times"] & H^n(X \times Y, A \times Y, R)
\end{tikzcd}$$

$[\varphi] \in H^i(Z, B, R)$

$[\psi] \in H^{n-i}(Y, R)$

Clearly the following commutes:

$$\begin{tikzcd}
X \times Y \arrow[r, "f \times id_Y"] \arrow[d, "p_X"] \arrow[rr, bend left, "p_Y"] \arrow[dr, phantom, "\circlearrowright"] & Z \times Y \arrow[d, "p_Z"] \arrow[dr, "p_Y"] \arrow[r, phantom, "\circlearrowright"] & {} \\
X \arrow[r, "f"] & Z & Y
\end{tikzcd}$$

$$\begin{aligned}
(f \times id)^* ([\varphi] \times [\psi]) &= (f \times id)^* \left(p_Z^* ([\varphi]) \cup p_Y^* ([\psi]) \right) \\
&= \left((f \times id)^* p_Z^* \right) ([\varphi]) \cup \left((f \times id)^* p_Y^* \right) ([\psi]) \\
&= \left(p_Z \circ (f \times id) \right)^* ([\varphi]) \cup \left(p_Y \circ (f \times id) \right)^* ([\psi]) \\
&= (f \circ p_X)^* ([\varphi]) \cup p_Y^* ([\psi]) \\
&= p_X^* (f^* ([\varphi])) \cup p_Y^* ([\psi]) \\
&= f^*([\varphi]) \times [\psi],
\end{aligned}$$

as desired.

Now we show commutativity of the following:

$$\begin{tikzcd}
H^i(A, R) \underset{R}{\otimes} H^{n-i}(Y, R) \arrow[r, "\times"] \arrow[d, "\partial \otimes id"] & H^n(A \times Y, R) \arrow[d, "\partial"] \\
H^i(X, A, R) \underset{R}{\otimes} H^{n-i}(Y, R) \arrow[r, "\times"] & H^n(X \times Y, A \times Y, R)
\end{tikzcd}$$

$[\varphi] \in H^i(A, R)$, $\quad \varphi: C_i(A, \mathbb{Z}) \longrightarrow R$, $\quad d(\varphi) = \varphi \circ d = 0$

$[\psi] \in H^{n-i}(Y, R)$, $\quad \psi: C_{n-i}(Y, \mathbb{Z}) \longrightarrow R$, $\quad d(\psi) = \psi \circ d = 0$

$\exists \bar{\varphi}: C(X, \mathbb{Z}) \longrightarrow R$ extension, i.e. $\bar{\varphi}|_{C_i(A, \mathbb{Z})} = \varphi$. (The existence simply follows from the involved modules being free.) (More generally, use the surjection in the LES.)

$d(\bar{\varphi}) = \bar{\varphi} \circ d \in C^{i+1}(X, A, R)$, $\quad d(C_{i+1}(A, \mathbb{Z})) \subseteq C_i(A, \mathbb{Z})$

$\bar{\varphi} \circ d|_{C_{i+1}(A, \mathbb{Z})} = \varphi \circ d = 0$

$dd(\bar{\varphi}) = 0 \implies \partial([\varphi]) = [d(\bar{\varphi})] = [\varphi \circ d] \in H^{i+1}(X, A, R)$

Goal: $\partial([\varphi]) \times [\psi] = \partial([\varphi] \times [\psi])$.

$$[\varphi]\times[\psi] = \left[p_A^*(\varphi) \cup p_Y^*(\psi)\right] = [\underbrace{\varphi\circ p_{A*} \cup \psi\circ p_{Y*}}]$$

this is a fuction $C_n(A\times Y, \mathbb{Z}) \to R$

where $p_{A*} : C_i(A\times Y, \mathbb{Z}) \longrightarrow C_i(A, \mathbb{Z})$,

$p_{Y*} : C_{n-i}(A\times Y, \mathbb{Z}) \longrightarrow C_{n-i}(Y, \mathbb{Z})$

$C_n(X\times Y, \mathbb{Z}) \longrightarrow R$ $\qquad \bar\varphi\circ p_{X*} \cup \psi\circ p_{Y*}$

$C_i(X\times Y, \mathbb{Z}) \xrightarrow{p_{X*}} C_i(X, \mathbb{Z}) \xrightarrow{\bar\varphi} R$ $\qquad \bar\varphi\circ p_{X*}$

$C_{n-i}(X\times Y, \mathbb{Z}) \xrightarrow[p_{Y*}]{} C_{n-i}(Y, \mathbb{Z}) \xrightarrow[\psi]{} R$ $\qquad \psi\circ p_{Y*}$

$$\bar\varphi\circ p_{X*} \cup \psi\circ p_{Y*}\Big|_{C_n(A\times Y, \mathbb{Z})} = \varphi\circ p_{A*} \cup \psi\circ p_{Y*}$$

$$\partial([\varphi]\times[\psi]) = d(\bar\varphi\circ p_{X*} \cup \psi\circ p_{Y*})$$

$$= \left[d\left(p_X^*(\bar\varphi) \cup p_Y^*(\psi)\right)\right]$$

Recall: $d(a\cup b) = da\cup b + (-1)^{|a|} a\cup db$

$$= \left[d\left(p_X^*(\bar\varphi) \cup p_Y^*(\psi)\right)\right]$$

$$= \left[d\left(p_X^*(\bar\varphi)\right) \cup p_Y^*(\psi)\right]$$

$$= \left[p_X^*(d(\bar\varphi)) \cup p_Y^*(\psi)\right]$$

$$= p_X^*\left([d(\bar\varphi)]\right) \cup p_Y^*([\psi]) = [d(\bar\varphi)]\times[\psi] = \partial[\varphi]\times[\psi]$$ □

Rmk. $h^*(X, A) \xrightarrow{\times} k^*(X, A)$ is a natural trf. of coh. ths.

What happens if $A = \emptyset$, $X = *$?

$$H^*(pt, R) \underset{R}{\otimes} H^*(Y, R) \xrightarrow{\times} H^*(pt\times Y, R)$$

Claim: this is an iso.

$$H^n(pt, R) = \begin{cases} R & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\text{LHS} = \bigoplus_n H^i(pt, R) \underset{R}{\otimes} H^{n-i}(Y, R) = R \underset{R}{\otimes} H^n(Y, R) \cong H^n(Y, R).$$

Topology II, lecture 6

Prop. Suppose $\alpha^*: E^* \longrightarrow F^*$ is a nat. trf. of (gen.) hlg. ths. on CW-pairs.

If $\forall n \in \mathbb{Z}$: $\alpha^n: E^n(pt) \longrightarrow F^n(pt)$ is an iso,

then $\forall n \in \mathbb{Z}$ $\forall (X,A)$: $\alpha^n: E^n(X,A) \longrightarrow F^n(X,A)$ is an iso.

Pf: Only for finite dimensional X. Induction on $\dim X$.

Case $\dim X = 0$: $X = \coprod_\beta pt_\beta$

$\forall n$:

$$\begin{tikzcd} E^n(X) \arrow[r, "\cong"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & F^n(X) \arrow[d, "\cong"] \\ \prod_\beta E^n(pt_\beta) \arrow[r, "\prod \alpha^n", "\cong"'] & \prod_\beta F^n(pt_\beta) \end{tikzcd}$$

$\Rightarrow E^n(X) \xrightarrow{\cong} F^n(X)$

Fact.
By assumptions and htp invariance we could show that:

$\forall n \in \mathbb{Z}$ $\forall m \geqq 0$ $E^n(D^m) \xrightarrow{\cong} F^n(D^m)$

Induction.
Assume that the statement is proven for $\dim X \leq m-1$, $m \geqq 1$.

Given now an X with $\dim X = m$, consider

$\emptyset = X^{-1} \subseteq X^0 \subseteq \ldots \subseteq X^{m-1} \subseteq X^m = X$.

Claim. $\forall n$ the map $E^n\left(\bigvee_\beta S^m_\beta, pt\right) \xrightarrow{\cong} F^n\left(\bigvee_\beta S^m_\beta, pt\right)$

Pf: $\left(\coprod_\beta D^m_\beta, \coprod_\beta S^{m-1}_\beta\right)$

$\coprod_\beta D^m_\beta \Big/ \coprod_\beta S^{m-1}_\beta \cong \bigvee_\beta S^m_\beta$

By the disj union axiom and the previous step with discs:

$$\begin{tikzcd} E^n\left(\coprod_\beta D^m_\beta\right) \arrow[r, "\alpha^n", "\cong"'] \arrow[d, "\cong"] & F^n\left(\coprod_\beta D^m_\beta\right) \arrow[d, "\cong"] \\ \prod_\beta E^n(D^m_\beta) \arrow[r, "\prod_\beta \alpha^n", "\cong"'] & \prod_\beta F^n(D^m_\beta) \end{tikzcd}$$

$\Rightarrow$ the top now is an iso.

Again by disj union ax. and the induction assumption:

$$\begin{tikzcd}
E^n\left(\coprod_\beta S^m_\beta\right) \arrow[r, "\alpha^n", "\cong"'] \arrow[d, "\cong"] & F^n\left(\coprod_\beta S^{m-1}_\beta\right) \arrow[d, "\cong"] \\
\prod_\beta E^n(S^m_\beta) \arrow[r, "\prod_\beta \alpha^n", "\cong"'] & \prod_\beta F^n(S^{m-1}_\beta)
\end{tikzcd}$$

$\Rightarrow$ the top now is an iso

By the les and naturality:

$$\begin{tikzcd}
E^{n-1}\left(\coprod_\beta D^m_\beta\right) \arrow[r] \arrow[d, "\alpha^{n-1}", "\cong"'] & E^{n-1}\left(\coprod_\beta S^{m-1}_\beta\right) \arrow[r, "\partial"] \arrow[d, "\alpha^{n-1}", "\cong"'] & E^n\left(\coprod_\beta D^m_\beta, \coprod_\beta S^{m-1}_\beta\right) \arrow[r] \arrow[d, "\cong"] & E^n\left(\coprod_\beta D^m_\beta\right) \arrow[r] \arrow[d, "\alpha^n", "\cong"'] & E^n\left(\coprod_\beta S^{m-1}_\beta\right) \arrow[d, "\alpha^n", "\cong"'] \\
F^{n-1}\left(\coprod_\beta D^m_\beta\right) \arrow[r] & F^{n-1}\left(\coprod_\beta S^{m-1}_\beta\right) \arrow[r] & F^n\left(\coprod_\beta D^m_\beta, \coprod_\beta S^{m-1}_\beta\right) \arrow[r] & F^n\left(\coprod_\beta D^m_\beta\right) \arrow[r] & F^n\left(\coprod_\beta S^{m-1}_\beta\right)
\end{tikzcd}$$

The middle vertical arrow is an iso by the 5-lemma.

The following commutes:

$$\begin{tikzcd}
E^n\left(\coprod_\beta D^m_\beta, \coprod_\beta S^{m-1}_\beta\right) \arrow[d, "\cong"'] & E^n\left(\bigvee_\beta S^m_\beta, pt\right) \arrow[l, "\text{canonical}", "\text{projection}^* \ \cong"'] \arrow[d, "\cong"] \\
F^n\left(\coprod_\beta D^m_\beta, \coprod_\beta S^{m-1}_\beta\right) & F^n\left(\bigvee_\beta S^m_\beta, pt\right) \arrow[l, "\cong"]
\end{tikzcd}$$

The horizontal maps are isos by excision, the left map is iso by the above. $\Rightarrow$ The right map is an iso, proving the Claim. $\boxed{C}$

<u>General case:</u>

dim $X = m$, $\emptyset = X_{-1} \subseteq \dots \subseteq X^m = X$

$X^m/X^{m-1} \cong \bigvee_\beta S^m_\beta$ Now we apply the les to (X^m, X^{m-1}):

$$\begin{tikzcd}
E^{n-1}(X^{m-1}) \arrow[r, "\partial"] \arrow[d, "\alpha^{n-1}"', "\cong"] & E^n(X^m, X^{m-1}) \arrow[r] \arrow[d, "\alpha^n\,(\cong)"] & E^n(X^m) \arrow[r] \arrow[d, "\alpha^n", "\cong"'] & E^n(X^{m-1}) \arrow[r, "\partial"] \arrow[d, "\alpha^n", "\cong"'] & E^{n+1}(X^m, X^{m-1}) \arrow[d, "\alpha^{n+1}", "\cong"'] \\
F^{n-1}(X^{m-1}) \arrow[r] & F^n(X^m, X^{m-1}) \arrow[r] & F^n(X^m) \arrow[r] & F^n(X^{m-1}) \arrow[r, "\partial"] & F^{n+1}(X^m, X^{m-1})
\end{tikzcd}$$

$$\begin{tikzcd}
E^n(X^m, X^{m-1}) \arrow[d, "\alpha^n", "\cong"'] & E^n(X^m/X^{m-1}, pt) \arrow[l, "\cong", "\text{by excision}"'] \arrow[d, "\cong \text{ by the Claim}", "\alpha^n"'] \\
F^n(X^m, X^{m-1}) & F^n(X^m/X^{m-1}, pt) \arrow[l, "\cong", "\text{by excision}"']
\end{tikzcd}$$

The middle arrow is an iso by the 5-lemma, proving the prop.

□

This proposition yields the Künneth theorem. □

<u>Cor.</u> There is a gr. ring iso $H^*(X,R) \underset{R}{\otimes} H^*(Y,R) \xrightarrow{\times} H^*(X\times Y, R)$

if X, Y are CW and $H^\ell(Y,R)$ is a fin. gen. projective R-module $\forall \ell$. □

Consequence: if $n_1, \dots, n_k$ are odd, then

$$H^*(S^{n_1} \times \dots \times S^{n_k}, R) \cong \underset{|\alpha_i| = n_i}{\Lambda_R} [\alpha_1, \dots, \alpha_k]$$

Recall that the RHS means $\alpha_i^2 = 0$, $\alpha_i\alpha_j = -\alpha_j\alpha_i$ $\forall i \neq j$.

This is false for non-odd n_i. ⟶ this will fail

E.g. $H^1(S^1 \times S^1, R) \cong H^1(S^1, R) \underset{R}{\otimes} H^1(S^1, R)$

$$\underset{|\alpha| = 1}{\Lambda_R[\alpha]} \underset{R}{\otimes} \underset{|\beta| = 1}{\Lambda_R[\beta]}$$

$\alpha^2 = 0$, $\beta^2 = 0$

$\alpha_1 := \alpha \otimes 1$, $\alpha_2 := 1 \otimes \beta$, $\alpha_1^2 = \alpha_2^2 = 0$

$\alpha_1\alpha_2 = (\alpha \otimes 1)\cdot(1 \otimes \beta) = \alpha \otimes \beta$

$\alpha_2\alpha_1 = (1 \otimes \beta)\cdot(\alpha \otimes 1) = (-1)^{|\alpha|\cdot|\beta|}\alpha \otimes \beta = -\alpha \otimes \beta$

<u>Rmk.</u> $S^3 \times S^3$ and $S^3 \vee S^3 \vee S^6$ cannot be distinguished by π_n, H_n, H^n.

But $H^*(S^3 \times S^3, \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1, \alpha_2]$ $|\alpha_1| = |\alpha_2| = 3$, $\alpha_1\alpha_2 \neq 0$

$H^*(S^3 \vee S^3 \vee S^6, \mathbb{Z})$ does not have nontrivi products $\Big\} \to S^3 \times S^3 \not\simeq S^3 \vee S^3 \vee S^6$

$\downarrow \cong$

$H^*(S^3) \times H^*(S^3) \times H^*(S^6)$ $\forall * > 0$

(It is clear that they are $\not\simeq$ by the wedge construction.)

Cap product

30.04.2018

Recall that we computed $H^*(S^{n_1}\times\dots\times S^{n_k};R)$ as a ring. ($\forall n_i$ odd)

For this we used the Künneth Theorem.

The method we will use is based on Poincaré duality.

Poincaré duality: M closed R-oriented mf., $\dim M = n$. Then there is a well-defined class $[M]\in H_n(M;R)$ (called the fundamental class if $R=\mathbb{Z}$) s.t.

$$[M]\cap - : H^k(M;R) \xrightarrow{\cong} H_{n-k}(M;R)$$

is an isomorphism.

The notions in the above statement will be explained in detail later.

Rmk. Betti, Poincaré : $\dim_{\mathbb{Q}} H_k(M;\mathbb{Q}) = b_k(M)$ k^{th} Betti number of M.

For $n=\dim M$: $b_{n-k}(M) = b_k(M)$.

UCT $\Rightarrow$ $H^k(M;\mathbb{Q}) \cong H_k(M;\mathbb{Q})^\vee$ where $\vee$ denotes the dual vct space

(I would really like to avoid overloading $*$ even more)

$\Rightarrow H_k(M;\mathbb{Q}) \cong H^k(M;\mathbb{Q})^\vee$. This yields $b_k(M) = b_{n-k}(M)$ for $R=\mathbb{Q}$.

Noether: relations b/w coh. gps in the background.

Given a top. space X, a ring R, $l\le k$ integers there is a pairing

$$H_k(X;R)\times H^l(X;R) \xrightarrow{\cap} H_{k-l}(X;R)$$

that is R-bilinear, so we can write it as

$$H_k(X;R)\underset{R}{\otimes} H^l(X;R) \xrightarrow{\cap} H_{k-l}(X;R).$$

We first define a product (R-bilin.)

$$C_k(X;R)\underset{R}{\otimes} C^l(X;R) \xrightarrow{\cap} C_{k-l}(X;R).$$

Recall that $C_*(X;R) = C_*(X;\mathbb{Z})\underset{\mathbb{Z}}{\otimes} R$.

Let $(\sigma: \Delta^k\to X)\in C_k(X;R)$ and $\varphi\in C^l(X;R) = \mathrm{Hom}(C_l(X;\mathbb{Z}),R)\cong \mathrm{Hom}_R(C_l(X;R),R)$.

$$\sigma\cap\varphi := \varphi(d_{front}^*(\sigma))\cdot d_{back}^*(\sigma) \in C_{k-l}(X;R).$$

Here $d_{front}: \Delta^l \hookrightarrow \Delta^k$, $(t_0,\dots,t_l)\mapsto(t_0,\dots,t_l,0,\dots,0)$ and $d_{back}: \Delta^{k-l}\hookrightarrow\Delta^k$, $(s_0,\dots,s_{k-l})\mapsto(0,\dots,0,s_0,\dots,s_{k-l})$

Note that $\varphi(d^*_{front}(\sigma)) \in R$ and $d^*_{back}(\sigma) = \sigma \circ d_{back} : \Delta^{k-l} \to R$ is a sing. simplex.

Now for a generic $c = \sum_\alpha r_\alpha \sigma_\alpha \in C_k(X;R)$:

$$c \cap \varphi = \left(\sum_\alpha r_\alpha \sigma_\alpha \right) \cap \varphi = \sum_\alpha r_\alpha (\sigma_\alpha \cap \varphi) \in C_{k-l}(X;R)$$

Thus we defined $c \cap \varphi$ for any c, φ.

1) $(c + c') \cap \varphi = c \cap \varphi + c' \cap \varphi$ by def.

2) $c \cap (\varphi + \varphi') = c \cap \varphi + c \cap \varphi'$ by def.

<u>Kronecker product</u> ("reminder")

$$\underset{\varphi}{C^l(X;R)} \otimes_R \underset{c}{C_l(X;R)} \xrightarrow{\langle -,- \rangle} R$$

$C_l(X;R) = C_l(X;\mathbb{Z}) \otimes_{\mathbb{Z}} R$ is a free R-module,

$C^l(X;R) = \mathrm{Hom}_R(C_l(X;R))$ is the dual module

} usually of infinite dimension

The <u>Kronecker product</u> just evaluates φ on c:

$$\langle \varphi, c \rangle = \varphi(c) \in R.$$

<u>Lemma.</u> $\forall c, c' \in C_l(X;R)$: $c = c'$ iff $\forall \varphi \in C^l(X;R)$: $\langle \varphi, c \rangle = \langle \varphi, c' \rangle$

Pf: Elementary linalg or fucanal.

The idea is that $\forall \sigma$ basis elt (i.e. sing. simplex) define

$\varphi_\sigma(\tau) := \delta_{\sigma\tau}$ (Kronecker delta).

Test against these φ_σ fuctions. □

<u>Prop.</u> (Cap — cup formula)

$k \geq l$, $c \in C_k(X;R)$, $\varphi \in C^l(X;R)$, $\psi \in C^{k-l}(X;R)$

Then the following holds: $\langle \psi, c \cap \varphi \rangle = \langle \varphi \cup \psi, c \rangle$.

Pf. Suffices to check for $c : \Delta^k \to X$ since these form a basis.

$$\langle \psi, d(\sigma) \rangle = \langle d(\psi), \sigma \rangle$$

↑ this should be d^* (stupid convention)

This follows from $\underset{\langle \psi, d(\sigma) \rangle}{\underset{\parallel}{\psi(d(\sigma))}} = \psi\left(\sum (-1)^i d_i(\sigma)\right) = \left(\sum (-1)^i \psi \circ d_i\right)(\sigma) = \underset{\langle d\psi, \sigma \rangle}{\underset{\parallel}{(d\psi)(\sigma)}}$

Using this elementary fact, we will prove the cap – cup formula.

$$\langle \psi, \sigma\cap\varphi\rangle = \psi(\sigma\cap\varphi) = \psi\left(\varphi\left(d^*_{\text{front}}(\sigma)\right) d^*_{\text{back}}(\sigma)\right)$$

$$= \varphi\left(d^*_{\text{frnt}}(\sigma)\right)\psi\left(d_{\text{back}}(\sigma)\right)$$

$$= (\varphi\cup\psi)(\sigma) = \langle\varphi\cup\psi, \sigma\rangle$$

□

Lemma. $c\in C_k(X;R)$, $\varphi\in C^\ell(X;R)$, $k\geqslant\ell$, $k-\ell\geqslant 1$:

$$d(c\cap\varphi) = (-1)^\ell\big(\underbrace{dc\cap\varphi}_{\text{degree } k-\ell-1} - c\cap d\varphi\big) \qquad \text{(Leibniz identity)}$$

Pf: We will prove that $\forall\psi\in C^{k-\ell-1}(X;R)$:

$$\langle\psi, d(c\cap\varphi)\rangle = \langle\psi, (dc\cap\varphi - c\cap d\varphi)(-1)^\ell\rangle.$$

This implies the statement of the Lemma.

$$\langle\psi, d(c\cap\varphi)\rangle = \langle d\psi, c\cap\varphi\rangle \qquad \text{formula in the above proof}$$

$$= \langle\varphi\cup d\psi, c\rangle \qquad \text{cap–cup formula}$$

Leibniz: $d(\varphi\cup\psi) = d\varphi\cup\psi + (-1)^\ell\varphi\cup d\psi \Rightarrow \varphi\cup d\psi = (-1)^\ell\big(d(\varphi\cup\psi) - d\varphi\cup\psi\big)$

$$\Rightarrow \langle\psi, d(c\cap\varphi)\rangle = \langle(-1)^\ell(d(\varphi\cup\psi) - d\varphi\cup\psi), c\rangle$$

$$= (-1)^\ell\big(\langle d(\varphi\cup\psi), c\rangle - \langle d\varphi\cup\psi, c\rangle\big)$$

$$= (-1)^\ell\big(\langle\varphi\cup\psi, dc\rangle - \langle\psi, c\cap d\varphi\rangle\big)$$

$$= (-1)^\ell\big(\langle\psi, dc\cap\varphi\rangle - \langle\psi, c\cap d\varphi\rangle\big)$$

$$= \langle\psi, (-1)^\ell(dc\cap\varphi - c\cap d\varphi)\rangle$$

□

Note. One of the references has a nice appendix on Poincaré duality.

$\langle -,-\rangle : C^\ell(X;R)\times C_\ell(X;R) \longrightarrow R$ induces a well-def'd pairing

$$H^\ell(X;R)\times H_\ell(X;R) \longrightarrow R$$

$$[\varphi] \qquad [c] \qquad \langle\varphi, c\rangle = \varphi(c)$$

NTS $\varphi(c)$ is independent of representatives.

$d : C_\ell(X,\mathbb{Z}) \longrightarrow C_{\ell-1}(X,\mathbb{Z})$ induces $d \overset{\text{AON}}{=} d\otimes 1 : C_\ell(X;R) \to C_{\ell-1}(X;R)$

φ a cocycle, $\varphi\circ d = d(\varphi) = 0$ (we should use $d^*(\varphi)$ here)

c a cycle, $d(c) = 0$

$c - c' = d(c'')$

$$\langle \varphi, c\rangle = \langle \varphi, c' + d(c'')\rangle = \langle \varphi, c'\rangle + \underbrace{\langle \varphi, d(c'')\rangle}_{=\langle \varphi\circ d,\, c''\rangle = \langle 0, c''\rangle = 0} = \langle \varphi, c'\rangle \quad \Rightarrow \text{ independence of } c$$

Similarly: independence of φ.

__Rmk.__ We have already seen $\langle -,-\rangle : H^\ell(X;R) \otimes_R H_\ell(X;\underset{\mathbb{Z}}{R}) \longrightarrow R$ in the UCT: the adjoint of $\langle -,-\rangle$ is $H^\ell(X;R) \longrightarrow \mathrm{Hom}_R(H_\ell(X;\underset{\mathbb{Z}}{R}), R)$.

(The version with $\mathbb{Z}$ is the one in UCT.)

$\cap : C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R)$ $\quad k \geq \ell$ induces a well-def'd bilinear map $H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R)$, i.e. a map on the tensor product:

$$H_k(X;R) \otimes_R H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R)$$

$[c] \in H_k(X;R)$, $[\varphi] \in H^\ell(X;R)$, i.e. $dc = 0$, $d\varphi = \varphi\circ d = 0$.

Independence of $[c\cap\varphi]$ of choices:

$c = c' + dc'' \overset{\text{NTS}}{\Rightarrow} [c\cap\varphi] = [c'\cap\varphi]$, $\quad$ same for φ. (exercise)

$c\cap\varphi = c'\cap\varphi + \underbrace{dc''\cap\varphi}$. $\quad$ Claim: $dc''\cap\varphi$ is a bdry.

$$d((-1)^\ell c''\cap\varphi) \underset{\text{Leibniz}}{=} \underbrace{(-1)^{\ell+\ell}}_{1} dc''\cap\varphi - c''\cap\underbrace{d\varphi}_{0} = dc''\cap\varphi. \quad \Rightarrow [c\cap\varphi] = [c'\cap\varphi].$$

__Summary:__ X space, R comm. ring

$$H^k(X;R) \otimes_R H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R)$$

$$H_k(X;R) \otimes_R H^\ell(X;R) \xrightarrow{\cap} H_{k-\ell}(X;R)$$

$$H^\ell(X;R) \otimes_R H_\ell(X;R) \xrightarrow{\langle -,-\rangle} R$$

These are related by the cap-cup formula:

$$\langle [\psi]\cup[\varphi], [c]\rangle = \langle [\varphi], [c]\cap[\psi]\rangle$$

Prop. $f: X \longrightarrow Y$ cont map, $c \in C_k(X;R)$, $\varphi \in C^l(Y;R)$ or directly on (co)homology:

$[c] \in H_k(X;R)$ and $[\varphi] \in H^l(X;R)$, $k \geq l$.

$$\Rightarrow \quad f_*(c) \cap \varphi = f_*(c \cap f^*(\varphi)).$$

Pf: Easy exercise. □

There are also several relative versions of $\cap$.

$$H_k(X,A;R) \underset{R}{\otimes} H^l(X;R) \xrightarrow{\cap} H_{k-l}(X,A;R)$$

$$H_k(X,A;R) \underset{R}{\otimes} H^l(X,A;R) \xrightarrow{\cap} H_{k-l}(X;R)$$

Note that here we have $H(X;R)$ and not $H(X,A;R)$ on the right!

Most important one: X CW, $A,B \subseteq X$ subcomplexes or open subsets.

$$H_k(X,A\cup B;R) \underset{R}{\otimes} H^l(X,A;R) \xrightarrow[\text{excision}]{\cap} H_{k-l}(X,B;R)$$

02.05.2018

Topological manifolds, orientation and homology

Def. An n-dim <u>(topological) manifold</u> is a top. space M such that

1) $\forall x \in M$ $\exists U \subseteq M$, U open, $U \cong \mathbb{R}^n$ homeo.
2) M is Hausdorff.

Note that we do not require any smooth structure, second countability or connectedness.

Ex. S^n, $\mathbb{R}^n$, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, S_g  (genus g surface)

Prop. Given an m-dim mf. M and an n-dim mf. N, the space $M \times N$ is an $(m+n)$-dim mf.

Pf. Trivial. □

We cannot drop the Hausdorff condition:

$\mathbb{R} \times \{\pm 1\} \big/ (x,1) \sim (x,-1)$ if $x \neq 0$ 

$(0,1)$ and $(0,-1)$ cannot be separated

Topology II, lecture 8

Prop. Let M be an n-dim mf. and $x \in M$. Then

$$H_i(M, M\setminus x; R) \cong \begin{cases} R & i=n \\ 0 & \text{else} \end{cases}$$

Pf. Using classical form of excision: given $A, B \subseteq X$, $\overline{B} \subseteq \mathring{A}$ (no openness or closedness is required) $\Rightarrow \forall i$: $H_i(X\setminus B, A\setminus B; R) \xrightarrow[\text{canonical}]{\cong} H_i(X, A; R)$.

$X := M$, $A := M\setminus x$, $x \in U \cong \mathbb{R}^n$ an open nbh., $B := M\setminus U$. $\Rightarrow$ $\overline{B} \subseteq \mathring{A} = A$

Excision: $H_i(M, M\setminus x; R) \cong H_i(U, U\setminus x; R)$

Wma $x \to 0$ under $U \xrightarrow{\cong} \mathbb{R}^n$

$\Rightarrow$ $H_i(M, M\setminus x; R) \cong H_i(\mathbb{R}^n, \mathbb{R}^n\setminus 0; R)$

$\mathbb{R}^n$ is contractible $\Rightarrow$ it has $\tilde{H}_* = 0$.

LES $\Rightarrow$ $H_i(M, M\setminus x; R) \cong \tilde{H}_{i-1}(\mathbb{R}^n\setminus 0; R)$

$\mathbb{R}^n\setminus 0 \simeq S^{n-1}$ htp. equivalence (def. retract)

$\Rightarrow$ $H_i(M, M\setminus x; R) \cong \tilde{H}_{i-1}(S^{n-1}; R) = \begin{cases} R & i=n \\ 0 & \text{else} \end{cases}$ $\forall n \geq 1$.

Note that the Prop. holds for $n=0$ trivially. $\square$

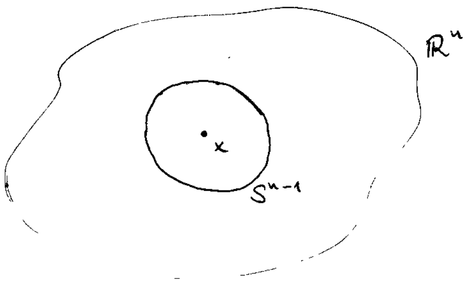
From now on we will use $R = \mathbb{Z}$ coefficients unless stated otherwise, at least today; in the next lecture we will return to the general case.

Def. An <u>orientation</u> of $\mathbb{R}^n$ at point x is a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n\setminus x)$.

<u>Intuition:</u> $S^{n-1} \subseteq \mathbb{R}^n \setminus x$

$H_n(\mathbb{R}^n, \mathbb{R}^n\setminus x) \cong H_{n-1}(\mathbb{R}^n\setminus x) \cong H_{n-1}(S^{n-1})$

Reflections change the generator of $H_n(\mathbb{R}^n, \mathbb{R}^n\setminus x) \cong \mathbb{Z} = \langle \pm 1\rangle$, rotations preserve the generator.



If we fix an orientation of $\mathbb{R}^n$ at point x then we can also choose an orientation at any other point in a canonical way.

Let $x \in \mathbb{R}^n$, $H_n(\mathbb{R}^n, \mathbb{R}^n\setminus x) = \langle \alpha \rangle$.

Now for $y \in \mathbb{R}^n$ take an open ball B with $x, y \in B$.

$$\begin{tikzcd}
 & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus x) = \langle \alpha \rangle \\
\langle \bar{\alpha} \rangle = H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \arrow[ur, "\cong", "exc."'] \arrow[dr, "\cong", "exc."'] & \\
 & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus y) = \langle \tilde{\alpha} \rangle
\end{tikzcd}$$

$\exists ! \ \bar{\alpha} \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B)$ generator

$\exists ! \ \tilde{\alpha} \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus y)$ generator

This shows that $\mathbb{R}^n$ is orientable.

Note that $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus B) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$ for any open ball $B \subseteq \mathbb{R}^n$.

Def. M an n-dim mf. A local orientation of M at $x \in M$ is a generator of $H_n(M, M \setminus x)$.

If $\forall x \in M$ one chooses $\mu_x \in H_n(M, M \setminus x)$ then these generators cannot be expected to be compatible (whatever that would mean).

Def. M an n-dim mf. An orientation of M is a function

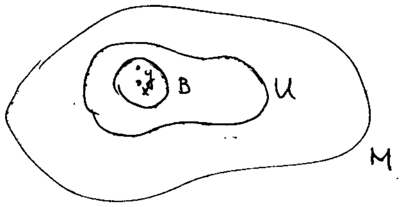
$$x \longmapsto \mu_x \in H_n(M, M \setminus x) \qquad \forall x \in M$$

such that $\forall x \in M \ \exists B \subseteq M$ open ball, $U \subseteq M$ open nbh. with

$$\begin{tikzcd}
H_n(M, M \setminus B) \arrow[r, "\cong"] & H_n(M, M \setminus y) & \forall y \in B. \\
\exists \ \mu_B \arrow[r, mapsto] & \mu_y &
\end{tikzcd}$$

i.e. there is a "universal" generator on B that restricts to the generator at any point.

Rmk. Excision: $H_n(M, M \setminus B) \cong H_n(U, U \setminus B)$
$\cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong \mathbb{Z}.$



Def. A manifold is orientable if it has an orientation.

Ex. S^2 is orientable, $\mathbb{R}P^2$ is not. (proof later)

Rmk. It will be an exercise that the choice of B is irrelevant.

Topology II, lecture 8

Orientation covering

Thm. For any n-dim mf. M there is a 2-sheeted covering space $\tilde{M} \xrightarrow{p} M$ such that $\tilde{M}$ is an orientable n-dim mf.

Rmk. Intuitively this means "untwisting" M.

Pf: $\tilde{M} := \{ \mu_x \mid x \in M,\ \mu_x \text{ is any generator} \}$ set

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{p} & M \\ \mu_x & \longmapsto & x \end{array} \quad \text{map}$$

As a set, $\tilde{M} = M \times \{\pm 1\}$ but not as spaces (we don't have a topology yet)

Neighbourhood bases for the topology on $\tilde{M}$:

given any $\mu_x \in \tilde{M}$ and B open ball in M, i.e. $B \subseteq U \cong \mathbb{R}^n$, $U \subseteq M$ open and $x \in B$, and $\langle \mu_B \rangle = H_n(M, M\setminus B)$, let

$$U(\mu_B) = \left\{ \vartheta \;\middle|\; \begin{array}{ccc} H_n(M, M\setminus B) & \xrightarrow{\cong} & H_n(M, M\setminus y) \\ \mu_B & \longmapsto & \vartheta \end{array} \quad y \in B \right\}$$

Denote the above map $H_n(M, M\setminus B) \longrightarrow H_n(M, M\setminus y)$ by res_y.

Then $U(\mu_B) = \{ \text{res}_y(\mu_B) \mid y \in B \}$.

Lemma. $U(\mu_B)$ form a neighbourhood basis (and hence they determine a topology on the set $\tilde{M}$).

Pf: Exercise, just check the axioms of a nbh. basis. $\boxed{2}$

The map $\tilde{M} \xrightarrow{p} M$ is now a covering map:

- if $B \subseteq M$ is an open ball then $p^{-1}(B) = U(\mu_B) \amalg U(-\mu_B)$.
- clearly $\begin{array}{ccc} U(\mu_B) & \xrightarrow{p|U(\mu_B)} & B \\ \text{res}_x(\mu_B) & \longmapsto & x \end{array}$ is a bijection. By our def. of the topology on $\tilde{M}$, it is a homeomorphism as well.

  Same for $p|U(-\mu_B) : U(-\mu_B) \xrightarrow{\cong} B$.

So p is a 2-sheeted covering map indeed.

Now comes the hard part.

Since $\tilde{M}$ is locally homeo. to M, it is locally homeo to $\mathbb{R}^n$; since a covering space of a T2 space is T2, it follows that $\tilde{M}$ is an n-dim mf. indeed.

For $\forall \mu_x \in \tilde{M}$ $\big(\mu_x \in H_n(M, M\setminus x),\ p(\mu_x) = x\big)$ we need an elt in $H_n(\tilde{M}, \tilde{M}\setminus \mu_x)$.

$\mu_x \in \mathcal{U}(\mu_B)$ for some open ball B and a generator μ_B,

$$\mu_B \in H_n(M, M\setminus B) \xrightarrow{res_x} \mu_x \in H_n(M, M\setminus x)$$

$$\begin{tikzcd}
H_n(\tilde{M}, \tilde{M}\setminus B) \arrow[r, phantom, "\underset{exc.}{\cong}"] & H_n(\mathcal{U}(\mu_B), \mathcal{U}(\mu_B)\setminus x) \arrow[r, phantom, "\underset{p_*}{\cong}"] & H_n(B, B\setminus x) \arrow[r, phantom, "\underset{exc.}{\cong}"] & H_n(M, M\setminus x) \\
\tilde{\vartheta} = \tilde{\mu}_x & & & & \vartheta = \mu_x \arrow[llll, mapsto]
\end{tikzcd}$$

So now we have $\tilde{M} \longrightarrow \{\tilde{\mu}_x \mid \tilde{\mu}_x \in H_n(\tilde{M}, \tilde{M}\setminus \mu_x)\}$;

$\mu_x \longmapsto \tilde{\mu}_x$

this is an orientation for $\tilde{M}$, as we will now prove.

1) $\tilde{\mu}_x$ does not depend on the choice of B

Suppose $x \in B_1 \cap B_2$ (open balls), $x \in \mathcal{U}(\mu_{B_1})$, $x \in \mathcal{U}(\mu_{B_2})$

$$\begin{tikzcd}
H_n(M, M\setminus B_1) \arrow[r, "\cong"] & H_n(M, M\setminus x) & H_n(M, M\setminus B_2) \arrow[l, "\cong"'] \\
\mu_{B_1} \arrow[r, mapsto] & \mu_x & \mu_{B_2} \arrow[l, mapsto]
\end{tikzcd}$$

$B \subseteq B_1 \cap B_2$ open ball

$$\begin{tikzcd}
\bullet & & & & & \mu_y \arrow[lllll, mapsto] \\
& H_n(\tilde{M}, \tilde{M}\setminus \mu_x) & H_n(\mathcal{U}(\mu_{B_1}), \mathcal{U}(\mu_{B_1})\setminus x) \arrow[l, "\cong", "exc."'] \arrow[r, phantom, "\cong"] & H_n(B_1, B_1\setminus x) \arrow[r, phantom, "\cong"] & H_n(M, M\setminus x) & \\
\bullet \arrow[uu, equal] & H_n(\tilde{M}, \tilde{M}\setminus \mu_x) \arrow[u, equal] & H_n(\mathcal{U}(\mu_B), \mathcal{U}(\mu_B)\setminus x) \arrow[l, "\cong"'] \arrow[u, "\cong"] \arrow[d, "\cong"'] \arrow[r, phantom, "\cong"] & H_n(B, B\setminus x) \arrow[u, "\cong"'] \arrow[d, "\cong"] \arrow[r, phantom, "\cong"] & H_n(M, M\setminus x) \arrow[u, equal] \arrow[d, equal] & \\
& H_n(\tilde{M}, \tilde{M}\setminus \mu_x) \arrow[u, equal] & H_n(\mathcal{U}(\mu_{B_2}), \mathcal{U}(\mu_{B_2})\setminus x) \arrow[l, "\cong"'] \arrow[r, phantom, "\cong"] & H_n(B_2, B_2\setminus x) \arrow[r, phantom, "\cong"] & H_n(M, M\setminus x) & \\
\bullet \arrow[uu, equal] & & & & & \mu_x \arrow[lllll, mapsto]
\end{tikzcd}$$

The whole diagram commutes, the isomorphisms come from excision.

2) $\mu_x \in \tilde{B}$. Need to find $\tilde{\mu}_{\tilde{B}} \in H_n(\tilde{M}, \tilde{M}\setminus \tilde{B})$ such that $\forall \vartheta \in \tilde{B}$:

$$H_n(\tilde{M}, \tilde{M}\setminus \tilde{B}) \ni \tilde{\mu}_{\tilde{B}} \xrightarrow{res_\vartheta} \tilde{\vartheta} \in H_n(\tilde{M}, \tilde{M}\setminus \vartheta)$$

$\mu_x \in \mathcal{U}(\mu_B) \Rightarrow \exists \tilde{B}$ open ball in $\tilde{M}$ s.t. $\mu_x \in \tilde{B} \subseteq \mathcal{U}(\mu_B)$

$\forall y \in p(\tilde{B}) \subseteq B$: $\forall \mu_y \in H_n(M, M\setminus y)$ generator: $\mu_y \in \tilde{B}$

$$\begin{tikzcd}
\text{generator } \mu_{\tilde{B}} \arrow[r, phantom, "\in"] \arrow[rrr, bend left, mapsto] & H_n(\tilde{M}, \tilde{M}\setminus\tilde{B}) \arrow[r, "\cong"] \arrow[d, no head, "\cong"'] & H_n(\tilde{M}, \tilde{M}\setminus\mu_y) \arrow[r, phantom, "\ni"] \arrow[d, no head, "\cong"', "\text{exc.}"] & \tilde{\mu}_y \\
 & H_n(U(\mu_B), U(\mu_B)\setminus\tilde{B}) \arrow[r, "\cong"] \arrow[d, no head, "\cong"', "p^*"] & H_n(U(\mu_B), U(\mu_B)\setminus\mu_x) \arrow[d, no head, "\cong"', "\text{exc.}"] & \\
 & H_n(B, B\setminus p(\tilde{B})) \arrow[r, "\cong"] \arrow[d, no head, "\cong"', "\text{exc.}"] & H_n(B, B\setminus y) \arrow[d, no head, "\cong"', "\text{exc.}"] & \\
 & H_n(M, M\setminus \underbrace{p(\tilde{B})}_{\text{ball}}) \arrow[r, "\cong"] & H_n(M, M\setminus y) & \\
 & H_n(M, M\setminus B) \arrow[u, "\text{canon.}", "\cong"'] \arrow[ur, "\cong", "\text{res}_y"'] & \mu_y \arrow[uuuur, bend right, mapsto] & \\
\mu_B \arrow[ur, phantom, "\in"] \arrow[uuuuu, mapsto] \arrow[urr, bend right, mapsto] & & &
\end{tikzcd}$$

$\mu_y \in \tilde{B} \subseteq U(\mu_B)$

$\Rightarrow$ $\tilde{M}$ is orientable.

□

Orientation character and R-orientability

07.05.2018

Recall: $\tilde{M} \xrightarrow{p} M$ 2-sheeted covering,

$$\tilde{M} = \{\alpha \in H_n(M, M\setminus x) \text{ generator } \mid x \in M\} \xleftrightarrow{1:1} \coprod_M \{\pm 1\}$$

$\tilde{M}$ is orientable with the topology given in the previous proof.

Exc. M orientable $\Longleftrightarrow$ $\exists s: M \to \tilde{M}$ section of p, i.e. $p \circ s = id_M$.

Consequence: sections $\xleftrightarrow{1:1}$ orientations.

$$s \longmapsto \{\underset{\text{generator}}{s(x) \in H_n(M, M\setminus x)} \mid x \in M\} =: \{\mu_x \mid x \in M\}$$

$$s := (s(x) = \mu_x) \longleftarrow \{\mu_x \mid x \in M\}$$

Prop. Suppose M is connected. Then $\tilde{M}$ has at most 2 path-components, and M is orientable iff $\tilde{M}$ has exactly 2 components.

Pf: Since M is a mf. (i.e. loc. euclidean and T2), it is loc. path-conn. Hence conn. and path-conn. components are the same; a mf. is the disjoint union of its path components. Thus the question of orientability may be reduced to the case of (path)-connected manifolds.

$\tilde{M} \xrightarrow[2]{p} M \quad \Rightarrow \quad \underline{\underline{\forall x \in M}}: \quad p^{-1}(x) = \{x_1, x_2\}$

WTS $\forall y \in \tilde{M}$ is in the (path-) component of x_1 or x_2

M path-conn. $\Longrightarrow$ $\exists \lambda : I \to M, \quad \lambda(0) = p(y), \quad y(1) = x$

Lift:

$$\begin{tikzcd} \tilde{M} \arrow[r, "p"] & M \\ I \arrow[u, dashed, "\tilde{\lambda}"] \arrow[ur, "\lambda"'] & \end{tikzcd}$$

$\tilde{\lambda}(0) = y$

$p\tilde{\lambda}(1) = \lambda(1) = x, \quad \tilde{\lambda}(1) \in p^{-1}(x) = \{x_1, x_2\} \rightarrow$ $\tilde{M}$ has at most 2 path-components

Suppose that $\tilde{M}$ has two path-components: $\tilde{M} = \tilde{M}_+ \amalg \tilde{M}_-$

where $\tilde{M}_+, \tilde{M}_-$ are (path-) conn. components.

A restriction of a covering space projection to a path-component is again a covering space, given that the base space is loc. path-connected, (which holds for M).

$\Rightarrow$ $p|_{\tilde{M}_-} : \tilde{M}_- \longrightarrow M, \quad p|_{M_+} : \tilde{M}_+ \longrightarrow M$ are $\overset{\text{distinct}}{\vee}$ covering spaces

Since p is two-sheeted, both of these restrictions must be 1-sheeted.

$\rightarrow$ they are homeomorphisms, and we have a section

$$\begin{tikzcd} M \arrow[rr, dashed] \arrow[dr, "\cong"'] & & \tilde{M} = \tilde{M}_+ \amalg \tilde{M}_- \\ & \tilde{M}_- \arrow[ur, hook, "\text{incl}"'] & \end{tikzcd}$$

$\Rightarrow$ M is orientable

(Or we could have said that $\tilde{M} \cong M \amalg M$ and path components of an orientable mf. are orientable.)

Suppose M is orientable. Then by previous discussion, we have

$s : M \longrightarrow \tilde{M}$ continuous section.

$\forall \alpha \in \tilde{M} \Rightarrow -\alpha \in \tilde{M}$

$\alpha \in H_n(M, M \setminus x)$ generator $\Rightarrow (-\alpha) \in H_n(M, M\setminus x)$ is also a gen.

$\Bigg\} \Rightarrow -s : M \longrightarrow \tilde{M}$

$x \longmapsto -s(x)$

is also a cont. section

Moreover, $-s$ is continuous since $-(\_)$ is:

$$-\mathcal{U}(\mu_B) = \mathcal{U}(-\mu_B).$$

So $s, -s$ are sections of a covering $\Rightarrow$ local homeos.

Local homeo. $\Rightarrow$ open, so $\tilde{M} = s(M) \amalg (-s)(M)$ is a decomposition into path-connected open sets. $\rightarrow$ $\tilde{M}$ has at most 2 components. □

<u>Prop.</u> There is a canonical $\mathbb{Z}/_2$-action on $\tilde{M}$ which is proper discontinuous. s.t.

$$\begin{tikzcd} \tilde{M} \arrow[r] \arrow[dr, "p"'] & \tilde{M}/(\mathbb{Z}/_2) \arrow[d, dashed, "\exists \cong"] \arrow[ld, phantom, "\circlearrowleft"] \\ & M \end{tikzcd}$$

Consequently, the deck transf. group $\underline{\mathrm{Deck}(\tilde{M}, p)} = \underline{\tilde{G}(\tilde{M}, p)} \cong \mathbb{Z}/_2$.
(This follows from cov. space theory.)

Pf: $\mathbb{Z}/_2 \cong \{+1, -1\}$

$$\left.\begin{array}{lll} +1 & \text{acts by identity,} & \tilde{M} \xrightarrow{id} \tilde{M} \\ -1 & \text{acts by} & \alpha \longmapsto -\alpha, \quad -(-\alpha) = \alpha \end{array}\right\} \mathbb{Z}/_2 \curvearrowright \tilde{M}$$

$\mathbb{Z}/_2$ is finite, $\tilde{M}$ is T2, the action is free (since α is a generator of $\mathbb{Z}$ and thus $\alpha \neq -\alpha$) $\Rightarrow$ $\mathbb{Z}/_2 \curvearrowright M$ in a proper discont. way
$\Rightarrow$ $\tilde{M} \xrightarrow{q} \tilde{M}/(\mathbb{Z}/_2)$ is a covering space, $\mathrm{Deck}(\tilde{M}, \mathbb{Z}/_2) \cong \mathbb{Z}/_2$.

By def of p, $p(\alpha) = p(-\alpha)$. Hence

$$\begin{tikzcd} \tilde{M} \arrow[r, "q"] \arrow[dr, "p"'] & \tilde{M}/(\mathbb{Z}/_2) \arrow[d, dashed, "\exists! h"] \\ & M \end{tikzcd}$$

$$\left.\begin{array}{l} \text{As } p \text{ and } q \text{ are 2-sheeted, } h \text{ is a cont. bijection.} \\ p \text{ and } q \text{ are open} \rightarrow h \text{ is open} \end{array}\right\} h \text{ is a homeomorphism.}$$

□

<u>Ex.</u> $M = \mathbb{R}P^2$, $\tilde{M} \cong S^2$ (will be trivial after solving the exercises)

$\mathbb{Z}/_2 \curvearrowright \tilde{M} \cong S^2$, $S^2/(\mathbb{Z}/_2) \cong \mathbb{R}P^2$ This is a reality check.

Prop. M connected $\Rightarrow$ we have an exact sequence

$$0 \longrightarrow \pi_1(\tilde{M}, \vartheta) \xrightarrow{p_*} \pi_1(M, x) \xrightarrow{w_1} \mathbb{Z}/_2$$

where $x \in M$, $\vartheta \in \tilde{M}$, $p(\vartheta) = x$, w_2 is called the orientability character.

Moreover, M is orientable iff $w_1 = 0$.

Pf: $\tilde{M} \xrightarrow{p} M$ is a normal (or Galois) covering, i.e. $p_* \pi_1(\tilde{M}, \vartheta) \triangleleft \pi_1(M, x)$, since every 2-sheeted covering is normal (because every index 2 subgroup is normal).

$$0 \longrightarrow \pi_1(\tilde{M}, \vartheta) \xrightarrow{p_*} \pi_1(M, x) \longrightarrow \mathrm{Deck}(\tilde{M}, p) \quad \text{exact}$$

The map on the right may not be surjective if $\tilde{M}$ is not connected.

We have $\mathrm{Deck}(\tilde{M}, p) \cong \mathbb{Z}/_2$, hence the exact sequence in the assertion above.

$$\begin{tikzcd}
\pi_1(M, x) \arrow[r] \arrow[dr, "w_1"'] & \mathrm{Deck}(\tilde{M}, p) \arrow[d, "\cong"] \\
 & \mathbb{Z}/_2
\end{tikzcd}$$

$$w_1([\gamma]) = \begin{cases} 0 & \text{if } \tilde{\gamma} \text{ is closed} \\ +1 & \text{if } \tilde{\gamma} \text{ is not closed} \end{cases}$$

where $\tilde{\gamma} : I \longrightarrow \tilde{M}$ is the unique lift of γ with $\tilde{\gamma}(0) = \vartheta$.

Note: in alg. top., w_1 is (equivalent to) the first Stiefel–Whitney class of M.

$$\underset{\substack{\rotatebox{90}{$\in$} \\ w_1}}{H^1(M, \mathbb{Z}/_2)} \cong \mathrm{Hom}\big(\underset{\substack{\rotatebox{90}{$\in$} \\ w_1}}{\pi_1(M, x)^{ab}}, \mathbb{Z}/_2\big)$$

Analogously one may define $w_n \in H^n(M, \mathbb{Z}/_2)$ $\longrightarrow$ Stiefel–Whitney classes

Back to the proof.

If M is ori. $\Rightarrow$ $\tilde{M}$ decomposes into a disj. union of 2 path-components which are homeo to M via p.

$C_\vartheta :=$ path-comp. of ϑ

$p|_{C_\vartheta} : \; C_\vartheta \xrightarrow{\cong} M$

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\tilde{M},\vartheta) \arrow[r, "\cong"] \arrow[rr, bend right, "p_* \;\cong"'] & \pi_1(C_\vartheta,\vartheta) \arrow[r, "(p|_{C_\vartheta})_*", "\cong"'] & \pi_1(M,x) \arrow[r, "w_1"] & \mathbb{Z}/2
\end{tikzcd}$$ exact

p_* is a composition of isos $\Rightarrow$ iso

Ker w_1 = Im p_*

$\Rightarrow w_1 = 0$ ✓

If $w_1 = 0$: again, let C_ϑ be the path-component of ϑ.

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\tilde{M},\vartheta) \arrow[r, "p_*"] \arrow[dr, "\cong"'] & \pi_1(M,x) \arrow[r, "w_1=0"] & \mathbb{Z}/2 \\
& & \pi_1(C_\vartheta,\vartheta) \arrow[u, "(p|_{C_\vartheta})_*"'] &
\end{tikzcd}$$

$\Rightarrow p_*$ is an iso

$\Rightarrow (p|_{C_\vartheta})_*$ is an iso. Since $p|_{C_\vartheta}$ is a covering space b/w path-connected spaces, it must be a homeomorphism.

Since $\tilde{M}$ is orientable, so is C_ϑ, and by homeomorphism, so is M. □

Cor. M connected. If $\pi_1(M)$ does not have an index 2 subgroup then M is orientable.

Pf: $\pi_1(M) \xrightarrow{w_1} \mathbb{Z}/2$ ↯ If $w_1 \neq 0$, it is surjective.

$\Rightarrow \mathrm{Ker}(w_1) \trianglelefteq \pi_1(M)$ is an index 2 subgroup. ↯ $w_1 = 0 \Rightarrow$ orientability. □

Cor. $\pi_1(M) = 1 \Rightarrow$ orientable.

Pf: 1 has no index 2 subgroup. □

Ex. S^n $(n \geqslant 2)$ are orientable.

One can check by hand that S^1 is also orientable, but this will also follow in the next lecture.

Ex. $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$ not orientable (exercise)

· $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$ but $\mathbb{RP}^3$ is orientable (later)

R-orientability

R a commutative ring (would work for ab. gps. too).

Important cases: $R = \mathbb{Z}$, $R = \mathbb{Z}/2$.

Given an n-manifold M, define

$$\underline{M_{\mathbb{Z}}} := \{\alpha \mid \alpha \in H_n(M, M\setminus x; \mathbb{Z}),\ x \in M\} = \coprod_{x\in M} H_n(M, M\setminus x; \mathbb{Z})$$

Now we can introduce a topology on $M_{\mathbb{Z}}$ the same way as we did for $\tilde{M}$.

$\forall x \in M$ let B be an open ball, $x \in B \subseteq U \subseteq M$, $U \cong \mathbb{R}^n$

$U(\alpha_B) := \{\operatorname{res}_x(\alpha_B) \mid x \in B\}$ for $\alpha_B \in H_n(M, M\setminus B; \mathbb{Z})$ arbitrary (can be 0)

where $H_n(M, M\setminus B; \mathbb{Z}) \xrightarrow{\operatorname{res}_x = (id)_*} H_n(M, M\setminus x; \mathbb{Z})$.

These $U(\alpha_B)$ form a nbh. basis for $M_{\mathbb{Z}}$, making it a top. space.

$$p_{\mathbb{Z}} : M_{\mathbb{Z}} \longrightarrow M$$
$$\alpha \in H_n(M, M\setminus x; \mathbb{Z}) \mapsto x$$

Exercise: $M_{\mathbb{Z}} \cong M_0 \amalg M_{\pm 1} \amalg M_{\pm 2} \amalg \dots$ s.t. $M_0 \cong M$, $\forall M_{\pm k} \cong \tilde{M}$.

This is not surprising as $M_{\pm k} = \{\alpha \mid \alpha = \pm k \text{ in } H_n(M, M\setminus x; \mathbb{Z})\}$,

$M_0 = \{\alpha \mid \alpha = 0 \text{ in } H_n(M, M\setminus x; \mathbb{Z})\}$

and $M_{\pm 1} = \tilde{M}$ by def. of $\tilde{M}$.

$$\begin{tikzcd} M_{\pm k} \arrow[r, "\cong"] & \tilde{M} = M_{\pm 1} \\ \alpha \arrow[r, mapsto] & \frac{\alpha}{k} \end{tikzcd}$$

Rmk. M ori $\iff$ $\exists\, s : M \longrightarrow M_{\mathbb{Z}}$ cont section of $p_{\mathbb{Z}}$ s.t. $s(M) \subseteq M_{\pm 1}$.

$$\begin{array}{ccccc} H_n(M, M\setminus x; R) & \cong & H_n(M, M\setminus x; \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} R & \cong & R \\ \pm r & & \pm 1 \otimes r & & \end{array}$$

where $\pm 1 \in H_n(M, M\setminus x; R) \cong \mathbb{Z}$

One needs to keep in mind that R may have several units, all of which are generators.

$$M_R = \{\alpha \mid \alpha \in H_n(M, M\setminus x; R),\ x \in M\} \cong \coprod_{x\in M} H_n(M, M\setminus x; R)$$

Topology on M_R: same way as for $M_\mathbb{Z}$.

$U(\alpha_B) = \{res_x(\alpha_B) \mid x \in M\}$, $\alpha_B \in H_n(M, M\setminus B; R)$, B open ball around x,

$$H_n(M, M\setminus B; R) \xrightarrow{id_*} H_n(M, M\setminus x; R).$$

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$M_{\pm r} = \{\alpha \in M_R \mid \alpha = \pm r \text{ in } H_n(M, M\setminus x; R),\ x \in M\}$ for $r \in R^\times$ (i.e. a generator)

We topologise these sets:

$U(\alpha_B) = \{res_x(\alpha_B) \mid x \in B\}$ $\alpha_B \in H_n(M, M\setminus x; R)$, B open ball

$\hookrightarrow$ neighbourhood basis

This yields a projection
$$\begin{array}{ccc} M_R & \xrightarrow{p} & M \\ \alpha \in H_n(M, M\setminus x; R) & \longmapsto & x \end{array}$$
which turns out to be a covering map.

<u>Def.</u> M an n-mnf. Then M is <u>R-orientable</u> if $\exists\, s: M \to M_R$ continuous section of p (i.e. $p\circ s = id_M$) s.t. $s(M) \subseteq M_{\pm r}$ for some $r \in R^\times$.

If $2r = 0$, i.e. $r = -r$, then $M_{\pm r} \cong M$

If $2r \neq 0$ then $M_{\pm r} \cong \tilde{M}$.

$$M_R \cong \left(\coprod_{2r=0} M_{\pm r}\right) \amalg \left(\coprod_{2r\neq 0} M_{\pm r}\right) \cong \left(\coprod_{2r=0} M\right) \amalg \left(\coprod_{2r\neq 0} \tilde{M}\right)$$

Prop. M an n-mf., wlog M is connected (one can always work on components).

i) If char $R = 2$ then M is R-orientable

ii) If char $R \neq 2$ then M is ($\mathbb{Z}$-) orientable iff it is R-orientable.

Thus the only rings that actually matter are $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$.

Pf: i) Suppose char $R = 2$, $2 \cdot 1 = 0$

$\Rightarrow M_{\pm 1} \cong M$ given by $p|_{M_{\pm 1}} : M_{\pm 1} \longrightarrow M$. Let $s := \left(p|_{M_{\pm 1}}\right)^{-1}$.

$\Rightarrow M \xrightarrow[s]{} M_{\pm 1} \subseteq M_R$ section $\Rightarrow$ R-orientability.

ii) char $R \neq 2$. $\Rightarrow \exists r \in R^{\times}$, $2r \neq 0$.

$p : M_R \longrightarrow M$ can be restricted: $p|_{M_{\pm r}}$:

$$\begin{tikzcd} M_{\pm r} \arrow[r] & M \\ \tilde{M} \arrow[u, "\cong"] \arrow[ur, "p"'] \arrow[ur, phantom, shift left=3, "\circlearrowright"] & \end{tikzcd}$$

If s is a section for p: $M \underset{p}{\overset{s}{\rightleftarrows}} \tilde{M}$ then we can define a section for M_R:

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[rrr, bend right] & \tilde{M} \cong M_{\pm r} & \subseteq & M_R \end{tikzcd}$$

, which is indeed a section for $p_R : M_R \longrightarrow M$. .

Conversely, if $s' : M \to M_R$ is a section ($s'(x)$ is a generator of $H_n(M, M\setminus x; R)$ $\forall x \in M$) then

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[dr] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & M_{\pm r} \subseteq M_R \arrow[d, "\cong"] \\ & \tilde{M} \end{tikzcd}$$

□

Def'. M is <u>R-orientable</u> if $\exists s : M \longrightarrow M_R$ is a section of $p_R : M_R \longrightarrow M$ s.t. $s(x) \in H_n(M, M\setminus x; R)$ is a generator $\forall x \in M$.

Exercise: Def. and Def'. are equivalent under connectedness

Topology II, lecture 10

Fundamental classes

Thm. Let M be an n-dim mf., compact, connected; R a comm. rng.

1) If M is R-orientable then the restriction map (induced by the identity)
$$res_x : H_n(M;R) \longrightarrow H_n(M, M\setminus x; R) \cong R$$
is an isomorphism.

2) If M is non-R-orientable then
$$res_x : H_n(M;R) \longrightarrow H_n(M, M\setminus x; R) \cong R$$
is injective with image $\{r \in R \mid 2r = 0\}$.

3) $\forall i > n$: $H_i(M;R) = 0$.

Lemma. Let M be an n-manifold and suppose $K \subseteq M$ a compact subspace.

Then (1) $\forall i > n$ $H_i(M, M\setminus K; R) = 0$

(2) $\alpha \in H_n(M, M\setminus K; R)$. Then $\alpha = 0$ iff $res_x(\alpha) = 0 \in H_n(M, M-x; R)$ $\forall x \in K$.

(3) Given a cont. section
$$\begin{tikzcd} M \arrow[r] & M_R \\ x \arrow[r, mapsto] & \alpha_x \end{tikzcd}$$
$\exists ! \ \alpha \in H_n(M, M\setminus K; R)$ s.t. $res_x(\alpha) = \alpha_x$.

PF OF THM: We use the yet-to-be-proven Lemma in the case $K = M$:

(1) $\Rightarrow$ 3) : $H_i(M;R) = H_i(M, M\setminus M; R) = 0$ $\forall i > n$ ✓

$\Gamma_R(M) = \{ s : M \longrightarrow M_R$ continuous section, i.e. $p_R \circ s = id_M \}$

Pointwise addition and scalar multiplication makes $\Gamma_R(M)$ into an R-module.

$$\begin{tikzcd}
H_n(M;R) \arrow[r, "\gamma"] & \Gamma_R(M) \arrow[r, "ev_x"] & H_n(M, M\setminus x; R) \quad \forall x \in M \\
\alpha \arrow[r, mapsto] & (y \mapsto res_y(\alpha)) & \\
 & s \arrow[r, mapsto] & s(x) \\
\alpha \arrow[rr, mapsto] & & res_x(\alpha)
\end{tikzcd}$$

To see injectivity of ev_x, we use the connectedness assumption.

$\forall s, s' \in \Gamma_R(M)$, $ev_x(s) = ev_x(s') \Rightarrow s(x) = s(x') \Rightarrow s = s'$
$\forall x \in M$

(3) $\Rightarrow$ γ is an iso

Proving 1): M R-ori $\Rightarrow$ $\exists s: M \to M_R$ s.t. $s(y) \in H_n(M, M\setminus y; R)$ is a generator.

$ev_x(s) = s(x) \in H_n(M, M\setminus x; R)$ is a generator

$\Rightarrow$ ev_x is an iso if M is R-ori. ✓

Proving 2): M not R-ori $\Rightarrow$ any $s: M \to M_R$ section:

$s(M) \subseteq M_{\pm r}$, $2r = 0$ $\Rightarrow$ $ev_x(s) = s(x)$, $2s(x) = 0$ in $H_n(M, M\setminus x; R)$.

$\forall r$ with $2r = 0$: need to find a section $s: M \to M_R$ s.t.

$$ev_x(s) = \underbrace{\pm 1 \otimes r}_{\in H_n(M, M\setminus x; R) \cong R} \longleftrightarrow r$$

$$\begin{tikzcd} H_n(M, M\setminus x; \mathbb{Z}) \otimes R \arrow[r, "\cong"] & H_n(M, M\setminus x; R) \\ \pm \underbrace{\mu_x}_{\text{generator}} \otimes r \arrow[r, mapsto] & \pm r \end{tikzcd}$$

$\mu_x \otimes r = -\mu_x \otimes r$ because $2r = 0$.

$$\Rightarrow \quad \begin{tikzcd} M \arrow[r, "s_r"] & M_R \\ x \arrow[r, mapsto] & \pm \mu_x \otimes r \end{tikzcd}$$

provides a section with $ev_x(s_r) = r$. □

PF OF LEMMA: (1) Mayer–Vietoris argument: $K = K_1 \cup K_2$, $K, K_1, K_2 \subseteq M$ are cpt.

Suppose that the Lemma holds for $K_1, K_2, K_1 \cap K_2$. Then the Lemma holds for $K = K_1 \cup K_2$.

Relative MV: $X \supseteq U, V$ opens Then we have a long exact sequence:

$$\dots \to H_i(X,U) \oplus H_i(X,V) \to H_i(X, U \cup V) \xrightarrow{\partial} H_{i-1}(X, U \cap V) \to H_{i-1}(X,U) \oplus H_{i-1}(X,V) \to \dots$$

(Coefficients are dropped throughout the whole proof, we do everything over R.)

Topology II, lecture 10

Relative MV for $X = M$, $U = M \setminus K_1$, $V = M \setminus K_2$, $K = K_1 \cup K_2$

$$\dots \to H_i(M, M\setminus K_1 \cap K_2) \xrightarrow{\partial} H_i(M, M \setminus K) \to H_i(M, M\setminus K_1) \oplus H_i(M, M \setminus K_2) \to H_i(M, M \setminus K_1 \cap K_2) \xrightarrow{\partial} \dots$$

(For $A \subseteq B \subseteq M$ we have $H_i(M, M \setminus B) \xrightarrow[\operatorname{res}_A^B]{\operatorname{res}_A} H_i(M, M \setminus A)$)

$$\gamma \longmapsto (\operatorname{res}_{K_1}(\gamma), \operatorname{res}_{K_2}(\gamma))$$

$$(\alpha, \beta) \longmapsto \operatorname{res}_{K_1 \cap K_2}(\alpha) - \operatorname{res}_{K_1 \cap K_2}(\beta)$$

$H_i(M, M \setminus K) \subset 0$ $\forall i > n$ by our assumptions and the LES.

$\forall \alpha \in H_n(M, M\setminus K)$: $\operatorname{res}_x(\alpha) = 0 \Rightarrow \alpha = 0$

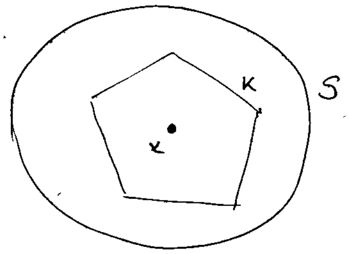
$\forall x \in K_1 \subset K$: $\operatorname{res}_x(\operatorname{res}_{K_1}(\alpha)) = \operatorname{res}_x(\alpha) = 0$ $\Rightarrow \operatorname{res}_{K_1}(\alpha) = 0$

$\forall y \in K_2 \subset K$: $\operatorname{res}_y(\operatorname{res}_{K_2}(\alpha)) = \operatorname{res}_y(\alpha) = 0$ $\rightarrow \operatorname{res}_{K_2}(\alpha) = 0$. $\Big\} \overset{MV}{\Rightarrow} \alpha = 0$

This proves that the MV argument holds for (1) and (2).

Case 1: $M = \mathbb{R}^n$. $K \subseteq \mathbb{R}^n$ is a cpt convex subset.

$\forall x \in K$: $\mathbb{R}^n \setminus x$ and $\mathbb{R}^n \setminus K$ both deformation retract on S, where S is a large enough sphere containing K



$\Rightarrow H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus x) \cong \tilde{H}_i(S)$

$\forall i > n$: $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K) = 0$.

If $\operatorname{res}_x(\alpha) = 0$ for some x $\Rightarrow$ $\alpha = 0$.

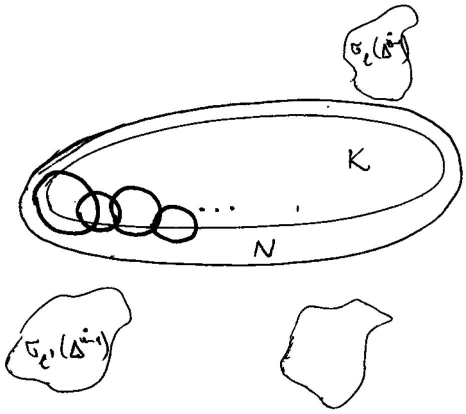
Case 2: $K = K_1 \cup \dots \cup K_r \subseteq \mathbb{R}^n = M$, $\forall K_i$ cx cpt

$\rightarrow$ follows from Case 1. using MV inductively.

Case 3: $K \subseteq \mathbb{R}^n$ any cpt.

$\forall \alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$, $\alpha = [\omega]$, $d\omega = \sum_e n_e \sigma_e$, $\forall \sigma_e : \Delta^{n-1} \longrightarrow \mathbb{R}^n$

$d\omega$ is in $\mathbb{R}^n \setminus K$, $\sigma_e(\Delta^{n-1}) \subseteq \mathbb{R}^n \setminus K$.



Choose a cpt. nbh. $N \supset K$ s.t.

$\forall \ell$: $\sigma_\ell(\Delta^{n-1}) \subseteq \mathbb{R}^n \setminus N$ still holds.

Cover K with $\overset{\text{closed}}{\text{balls}}$ $B_1, \dots, B_r$ finitely many,

$K \subseteq B_1 \cup \dots \cup B_r \subseteq N$, $\forall B_i \cap K \neq \emptyset$

$d\omega \in \mathbb{R}^n \setminus N \subseteq \mathbb{R}^n \setminus K$

$\alpha' := [\omega] \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus N)$, $res_K(\alpha') = \alpha$

Balls are convex compact, we like those, we can restrict from N to these balls, this is why we need N.

Goal: $H_i(M, M\setminus K) = 0$ $\forall i > n$, i.e. NTS $\alpha = 0$.

$$\begin{tikzcd}
H_i(M, M\setminus N) \arrow[rr, "res_x"] \arrow[dr] & & H_i(M, M\setminus K) \\
\alpha' \arrow[rr, mapsto] \arrow[ddr, mapsto] & \circlearrowright & \alpha \\
& H_i(M, M\setminus B_1\cup\dots\cup B_r) = 0 \arrow[uur] & \\
& res_{B_1\cup\dots\cup B_r}(\alpha') \arrow[uur, mapsto] &
\end{tikzcd}$$

By this factoring, $\alpha = 0$.

For $i = n$: $\forall \alpha \in H_n(M, M\setminus K)$: $res_x(\alpha) = 0$ $\forall x \in K$ $\Rightarrow$ $\alpha = 0$.

$$\begin{tikzcd}
H_n(M, M\setminus N) \arrow[r] & H_n(M, M\setminus B_1\cup\dots\cup B_r) \arrow[r] & H_n(M, M\setminus K) \\
\alpha' \arrow[r, mapsto] & res_{B_1\cup\dots\cup B_r}(\alpha') \arrow[r, mapsto] \arrow[dd, mapsto] & \alpha \\
& & =0 \\
& res_y(res_{B_1\cup\dots\cup B_r}(\alpha)) = 0 \arrow[r, Rightarrow, "\text{Case 2.}"] & res_{B_1\cup\dots\cup B_r}(\alpha) = 0 \arrow[u, Rightarrow] \\
& \forall y \in B_1 \cup \dots \cup B_r &
\end{tikzcd}$$

Case 4. M mf., $K \subseteq U \cong \mathbb{R}^n$, $U \subseteq M$ open

$$H_i(M, M\setminus K) \cong H_i(U, U\setminus K) \cong \underbrace{H_i(\mathbb{R}^n, \mathbb{R}^n\setminus K)}_{\text{we know what happens here by Case 3.}} \text{ by excision}$$

Case 5. M general, $K \subseteq M$ cpt.

$\Rightarrow$ $K = K_1 \cup \dots \cup K_r$, each K_i is as in Case 4. (Possible by cptness)

MV inductively reduces to Case 4.

Christian of course reformulates the Lemma:

(1) Same as before

(2)+(3) The canonical map

$$\begin{aligned} H_n(M, M\setminus x; R) &\longrightarrow \Gamma_A\begin{pmatrix} M_R \\ \downarrow \\ M \end{pmatrix} \text{ sections of } M_R \text{ defined on } A \\ \alpha &\longmapsto \left(x\in A \mapsto \text{res}_x(\alpha)\in H_n(M,M\setminus x;R)\right) \end{aligned}$$

is an iso.

It remains to prove surjectivity. We drop R from the notation.

Step 1. If K is suff. small (e.g. $K\subseteq B$ where B is a coordinate ball)

$$\begin{tikzcd} H_n(M, M\setminus B) \arrow[r, "\cong"] \arrow[d, "\text{canon.}"] & H_n(M, M\setminus x_0) \\ \Gamma_B\begin{pmatrix} M_R \\ \downarrow \\ M \end{pmatrix} \arrow[ur, hook, "ev_{x_0}\text{, injective since } B \text{ is connected}"'] & \end{tikzcd}$$

$\Rightarrow$ ev_{x_0} is an iso

$\Rightarrow$ the canon. map is an iso.

Step 2. $K = K_1 \cup K_2$, surj. holds for K_1 and K_2.

Use the rel. MV.

$$\longrightarrow 0 \longrightarrow H_n(M, M\setminus K) \xrightarrow{\binom{\text{res}}{\text{res}}} H_n(M, M\setminus K_1)\oplus H_n(M,M\setminus K_2) \xrightarrow{(\text{res}, -\text{res})} H_n(M, M\setminus K_1\cap K_2) \longrightarrow$$

Let $s \in \Gamma_K\begin{pmatrix} M_R \\ \downarrow \\ M \end{pmatrix}$ be a section.

By assumption, $\exists\, (\alpha_{K_1}, \alpha_{K_2}) \in H_n(M, M\setminus K_1) \oplus H_n(M, M\setminus K_2)$ s.t. $\text{res}_x\, \alpha_{K_i} = s_x$

$(\alpha_{K_1}, \alpha_{K_2})$ maps to 0 by injectivity of the canon. map for $K_1\cap K_2$.

$\Rightarrow$ $\exists\, \alpha_K$ s.t. $\text{res}_x\, \alpha_K = \left\{\begin{matrix} \text{res}_x\, \alpha_{K_1} & x\in K_1 \\ \text{res}_x\, \alpha_{K_2} & x\in K_2 \end{matrix}\right\} = s_x$

Step 3. $K\subseteq M$ arbitrary compact. We can write $K = K_1\cup\ldots\cup K_n$ where (3) holds for all K_i (e.g. use coordinate balls to obtain the K_i).

The statement follows using Step 2. inductively. □

We draw some consequences from the Theorem.

M is a connected compact n-manifold.

1) If M is orientable, $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and a generator $[\mu] \in H_n(M;\mathbb{Z})$ is called a fundamental class.

2) If M is non-orientable, $H_n(M;\mathbb{Z}) = 0$.

3) For any M we have $H_n(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$, i.e. there is a unique $\mathbb{Z}/2$-fundamental class.

Ex. $\mathbb{R}P^{2n}$ is non-orientable, $\mathbb{R}P^{2n+1}$ is orientable $\forall n$. This follows directly from the corollaries above.

Prop. M cpt conn n-mf. Then the torsion subgroup

$$\mathrm{Tors}(H_{n-1}(M;\mathbb{Z})) \cong \begin{cases} 0 & M \text{ ori} \\ \mathbb{Z}/2 & M \text{ non-ori} \end{cases}$$

Moreover, $H^n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & M \text{ ori} \\ \mathbb{Z}/2 & M \text{ non-ori} \end{cases}$

Pf: Use the UCT:

$$H_n\left(M;\mathbb{Z}/k\right) = \left(H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/k\right) \oplus \underbrace{\mathrm{Tor}\left(\mathbb{Z}/k, H_{n-1}(M;\mathbb{Z})\right)}_{\overset{\text{def}}{=} \mathrm{Tors}_k(H_{n-1}(M;\mathbb{Z}))}$$

If M ori, this yields

$$\mathbb{Z}/k = \mathbb{Z}/k \oplus \mathrm{Tors}_k(H_{n-1}(M;\mathbb{Z})) \Rightarrow \mathrm{Tors}_k(H_{n-1}(M;\mathbb{Z})) = 0. \ \forall k$$

$$\Rightarrow \mathrm{Tors}(H_{n-1}(M;\mathbb{Z})) = 0.$$

If M non-ori:

$$\left.\begin{matrix} \mathbb{Z}/2 & k \text{ even} \\ 0 & k \text{ odd} \end{matrix}\right\} = H_n(M;\mathbb{Z}/k) = 0 \oplus \mathrm{Tors}_k\left(H_{n-1}(M;\mathbb{Z})\right).$$

$$\Rightarrow \mathrm{Tors}(H_{n-1}(M;\mathbb{Z})) \underset{\substack{\uparrow \\ \forall k \text{ even}}}{=} \mathrm{Tors}_k(H_{n-1}(M;\mathbb{Z})) = \mathbb{Z}/2$$

The assertions about cohomology follow from the UCT for cohomology:

$$H^n(M;\mathbb{Z}) = \mathrm{Hom}\left(H_n(M;\mathbb{Z}), \mathbb{Z}\right) \oplus \mathrm{Ext}\left(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}\right)$$

$$\left.\begin{array}{lll}\text{Now if } M \text{ ori:} & H^n(M;\mathbb{Z}) = \mathbb{Z}\oplus 0. \\ \text{If } M \text{ non-ori:} & H^n(M;\mathbb{Z}) = 0 \oplus \mathbb{Z}/2. \end{array}\right\} (*)$$

□

Rmk. For the calculation of Ext in $(*)$ we need to know that $H_* M$ is a finitely generated group.

Note the nontriviality of this problem: the Whitehead problem asks if $\mathrm{Ext}(A,\mathbb{Z})=0$ implies that A is free, for an abelian group A.

This problem is undecidable. :(

We sketch the proof of $(*)$.

Compact manifolds are euclidean neighbourhood retracts (ENRs), i.e. we have an embedding $M^n \hookrightarrow \mathbb{R}^{n+k}$ and this M^n is a retract of an open nbhd in $\mathbb{R}^{n+k}$.

To see that this embedding exists:

If $M \supseteq U \cong \mathbb{R}^n$ is a coordinate chart then there is an associated collapse map

$$\begin{array}{rcl} M & \longrightarrow & (\mathbb{R}^n)^+ \cong S^n \\ U \ni m & \longmapsto & m \\ M\setminus U & \longmapsto & \infty \end{array}$$

(+ denotes the 1-pt compactification)

Cover M by fin. many charts (cptness)

$\Rightarrow M \hookrightarrow (S^n)^{\times m} \hookrightarrow \mathbb{R}^{(n+1)m}$ ✓

Fact 1. The retraction property is intrinsic to the embedded space and it is equivalent to weak local contractibility:

$\forall x \in M\ \exists U$ nbhd. $\exists V$ nbhd., $x\in V \subseteq U$ s.t. $V \hookrightarrow U$ is null-homotopic.

Fact 2. ENRs are retracts of finite simplicial complexes.

Now we relate the previous Prop. to so-called Bockstein sequences:

Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of abelian gps.

Then we have a short exact sequence of chain complexes

$$0 \longrightarrow C_*(X;A) \xrightarrow{f_*} C_*(X;B) \xrightarrow{g_*} C_*(X;C) \longrightarrow 0$$

This in turn induces a long exact sequence in homology, called the <u>Bockstein sequence</u>:

$$\cdots \longrightarrow H_{n+1}(X;C) \xrightarrow{\beta} H_n(X;A) \xrightarrow{f_*} H_n(X;B) \xrightarrow{g_*} H_n(X;C) \longrightarrow H_{n-1}(X;A) \longrightarrow \cdots$$

where β is called the <u>Bockstein homomorphism</u>.

Let M be a connected compact non-orientable n-mf.

Then $\mathrm{Tors}(H_{n-1}(M;\mathbb{Z}))$ can be recognised as the image of the Bockstein homomorphism associated to $0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$.

$$\begin{tikzcd}
\underbrace{H_n(M,\mathbb{Z})}_{0} \arrow[r] & \underbrace{H_n(M;\mathbb{Z}/2)}_{\cong \mathbb{Z}/2} \arrow[r, "\beta"] \arrow[dr, "\cong"'] & H_{n-1}(M;\mathbb{Z}) \arrow[r] & {} \\
 & & \mathrm{Tors} = \mathrm{Tors}_2 \arrow[u, phantom, "\cup|"] &
\end{tikzcd}$$

Now we move on to the non-compact case.

<u>Prop.</u> Let M be a non-compact connected n-mf.
Then $H_i(M;R) = 0$ $\forall i \geqslant n$ and $H_{n-1}(M;\mathbb{Z})$ is torsion-free, i.e. $\mathrm{Tors}(H_{n-1}(M;\mathbb{Z})) = 0$.

Pf: We drop R from the notation.

Let φ be a chain representing a class in $H_i(M)$ with compact img K.

Choose an open nbhd. $K \subseteq U$ where $\bar{U}$ is compact.

We can regard $[\varphi]$ as an elt in $H_i(U)$ $\Rightarrow$ it sts that $H_i(U) = 0$.

Consider the sequence associated to the triple $(M, \underbrace{U \amalg M\setminus\bar{U}}_{=M\setminus\partial U}, M\setminus\bar{U})$:

$$\begin{tikzcd}
{} \arrow[r] & \overset{0 \text{ for } i \geq n}{H_{i+1}(M, M\setminus\partial U)} \arrow[r] & H_i(U \amalg M\setminus\bar{U}, M\setminus\bar{U}) \arrow[r] & \overset{0 \text{ for } i>n}{H_i(M, M\setminus\bar{U})} \arrow[r] & {} \\
& & H_i(U) \arrow[u, "\cong", "\text{excision}"'] \arrow[r] & H_i(M) \arrow[u] & \\
& & {[\varphi]} \arrow[r, mapsto] & {[\varphi]} &
\end{tikzcd}$$

$\Rightarrow$ $H_i(U) = 0 \quad \forall i > n.$

It remains to deal with the case $i = n$.

Let $x_0 \in M\setminus\bar{U}$ (recall that M is non-cpt. and thus $\emptyset \neq M\setminus\bar{U}$)

$\mathrm{res}_{x_0}[\varphi] = 0$ in $H_n(M, M\setminus x_0)$ since $[\varphi]$ can be regarded as a class in $H_n(M\setminus x_0)$

$\Rightarrow$ the section $M \ni x \longmapsto \mathrm{res}_x[\varphi]$ vanishes by connectedness of M.

$\Rightarrow$ the img of $[\varphi]$ in $H_n(M, M\setminus\bar{U}) \cong \Gamma_{\bar{U}}\left(\begin{smallmatrix} M_R \\ \downarrow \\ M \end{smallmatrix}\right)$ vanishes

Looking at the diagram above we thus conclude that $[\varphi] = 0$ in $H_n(U)$.

Torsion freeness follows as before from the UCT. □